Math 105: Trigonometry and *fminsearch*

Objectives

- Relate sin() and cos() to unit circles
- Convert from rectangular to polar coordinates
- Calculate the position of a robotic arm (forward kinematics)
- Calculate the angles of a robotic arm (inverse kinematics)
- Use the Matlab function fminsearch()

Introduction

From Wikipedia,

> *Trigonometry (from Greek trigonon, "triangle" and metron, "measure"[1]) is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged in the Hellenistic world during the 3rd century BC from applications of geometry to astronomical studies. www.Wikipedia.com*

Trigonometry is fundamental to electrical and computer engineering.

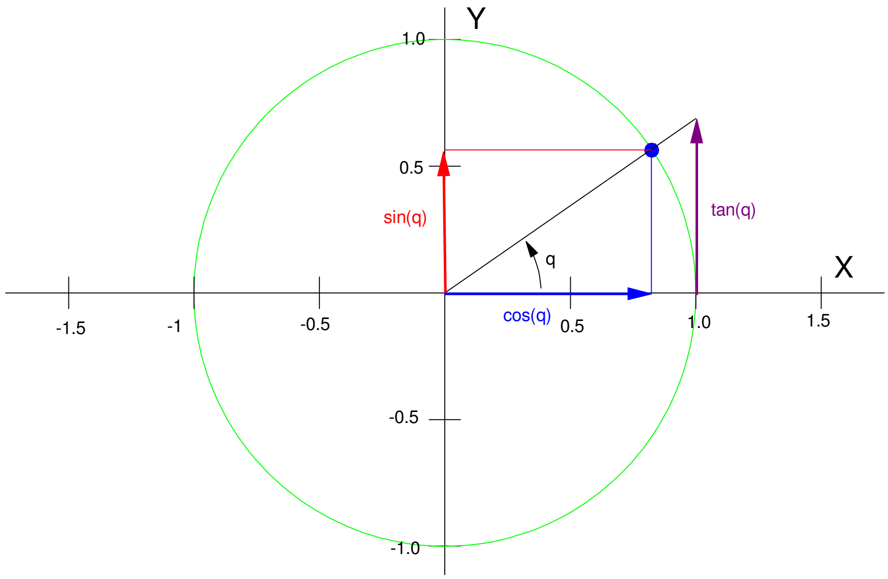
- Power is transmitted as a 60Hz sine wave
- Analysis of filters, such as the bass boost on your stereo, relies upon expressing an audio signal in terms of a sum of sinusoids,
- AC motors, such as a quad-copter motor, are driven by sinusoidal signals where the frequency of the sine wave determines the speed of the motor
- Analysis of systems described by differential equations (read: everything) depends upon being able to use complex numbers - which have their origin in sin() and cos() functions.

Likewise, trigonometry may seem like an archaic topic which deals only with architecture and triangles. Actually, it's much more than that.

sin(), cos(), tan()

Trigonometry is the study of the unit circle. If you draw a unit circle and take a point an angle θ from the x-axis, then

- The x-coordinate of that point is $\cos(\theta)$
- The y-coordinate of that point is $\sin(\theta)$
- If you extend the line from the origin to the point on the unit circle to x=1, the length of the line to the x-axis is $\tan(\theta)$

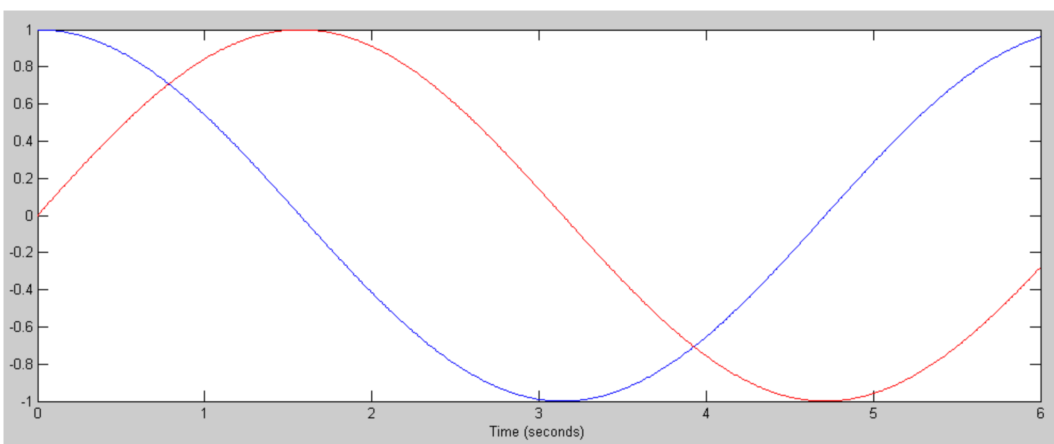


It you let the angle increase with time as

$$\theta = \omega t$$

then you get a sine wave. In Matlab:

```
>> t = [0:0.01:6]';
>> w = 1;
>> x = cos(w*t);
>> y = sin(w*t);
>> plot(t,x,t,y)
>> xlabel('Time (seconds)');
```



1 rad/sec sine wave: cos(t) (blue) and sin(t) (red)

Note that

- cos() and sin() go between -1 and +1. This isn't surprising since these are just the x and y coordinates as you go around the unit circle.
- The period of cos() and sin() is 2π (6.28 seconds).

The default units for cos() and sin() is radians. If you want to use degrees, the conversion is

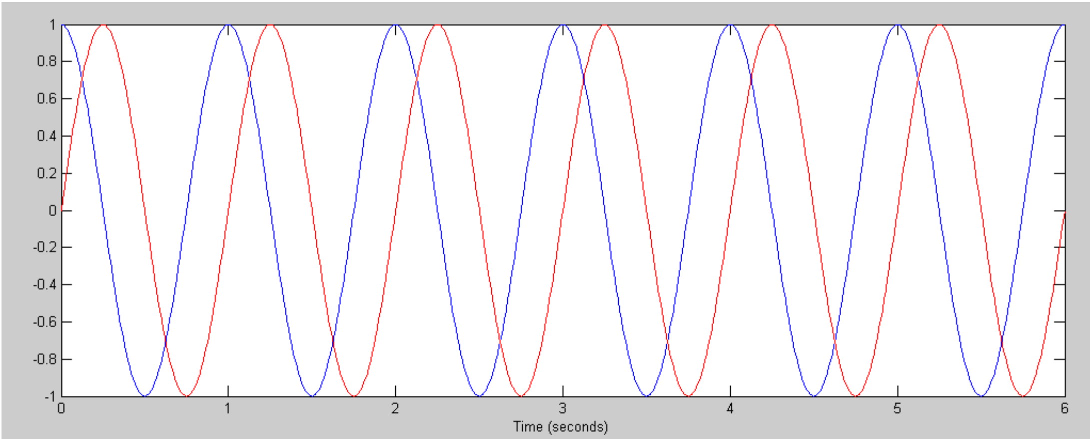
$$360 \text{ degrees} = 2\pi \text{ radians}$$

$$1\frac{\text{cycle}}{\text{second}} = 1Hz = 2\pi\frac{rad}{\text{sec}}$$

Pretty much, anything English isn't natural. You'll find in engineering that the math works out a lot nicer if you use natural units - such as radians.

If you increase the frequency, you get a sine wave that is quicker. A 1Hz sine wave $\left(2\pi\frac{\text{rad}}{\text{sec}}\right)$ looks like the following:

```
>> w = 2*pi;
>> x = cos(w*t);
>> y = sin(w*t);
>> plot(t,x,'b',t,y,'r')
>> xlabel('Time (seconds)');
```



1Hz Sine Wave: cos(6.28t) (blue) and sin(6.28t) (red)

Amplitude, Frequency, Phase

A generalized sine wave can be written as

$$y(t) = a\cos(\omega t) + b\sin(\omega t)$$

or

$$y(t) = r\cos(\omega t + \theta)$$

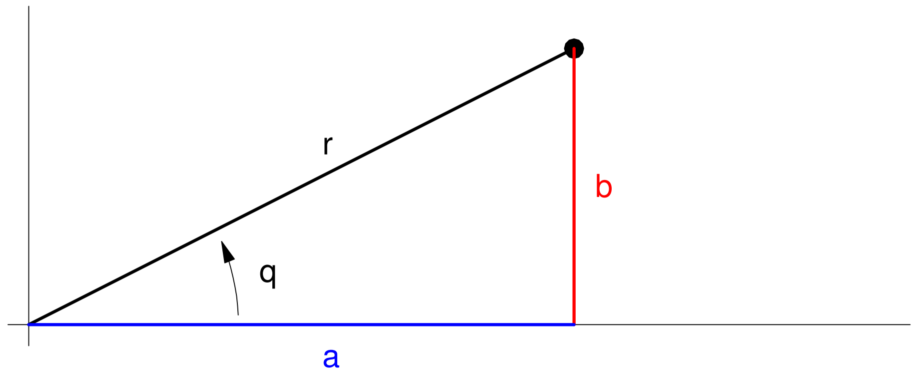
Here

- r is the amplitude
- ω is the frequency in rad/sec, and
- θ is the phase shift, also in radians.

The relationship between rectangular and polar form is

$$r^2 = a^2 + b^2$$

$$\tan(\theta) = \frac{b}{a}$$



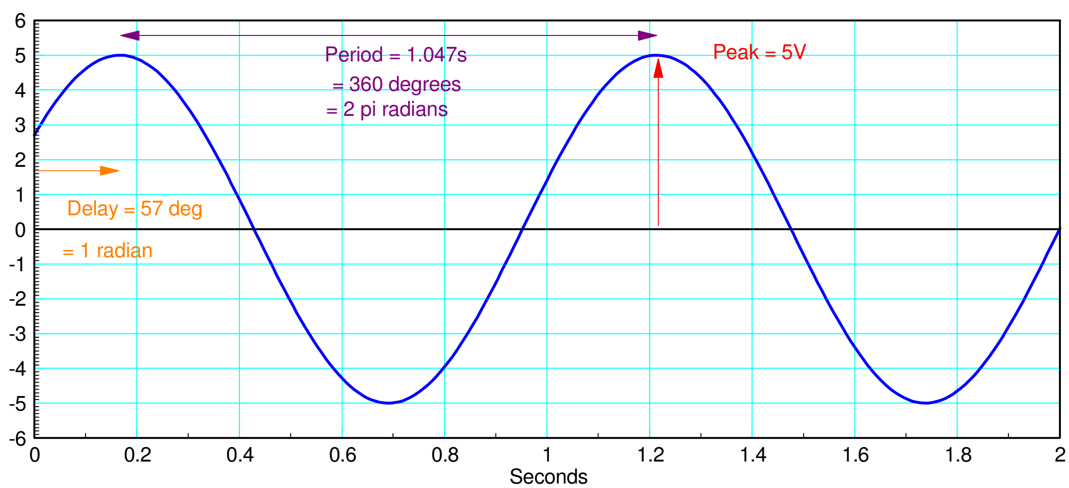
Conversion from rectangular form and polar form for sinusoids

For example,

$$y = 5\cos(6t - 1)$$

looks like the following:

```
>> t = [0:1/250:1]';
>> y = 5*cos(6*t-1);
>> plot(t,y);
```



Plot of y(t) = 5 cos(6t - 1)

The peak is 5 Volts

The frequency is 6 rad/sec, meaning the period is

- $period = 6T = 2\pi$
- $T = \frac{2\pi}{6} = 1.047 \text{ sec}$

The phase shift is 1 radian, meaning

- The delay is $\left(\frac{1 \text{ radian}}{6 \text{ rad/sec}}\right) = \frac{1}{6}\text{sec} \; = 0.166\,\text{sec}$

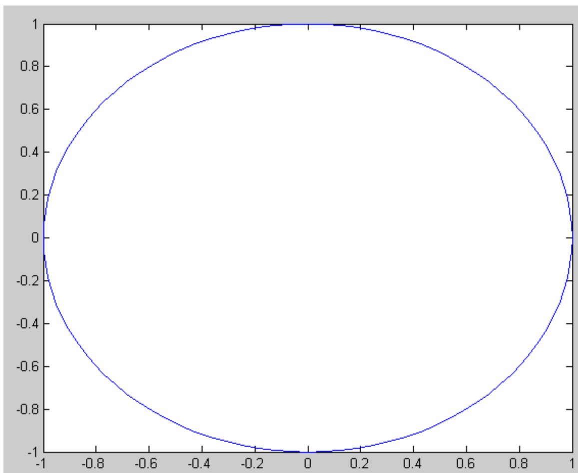
This is important in electrical and computer engineering:

- Given the amplitude, frequency, and phase shift, you can sketch a sine wave
- Given a sine wave, you can determine the amplitude, frequency, and phase shift.

Sine Waves and Circles

What shouldn't be surprising is that if you plot cos() vs sin() you get a circle

```
>> x = cos(w*t);
>> y = sin(w*t);
>> plot(x,y)
```



cos(t) vs. sin(t) produces the unit circle

It also shouldn't surprising that

$$\cos^2(t) + \sin^2(t) = 1$$

That just says that the circle you're using has a radius of one. That's sort of the definition of cos() and sin().

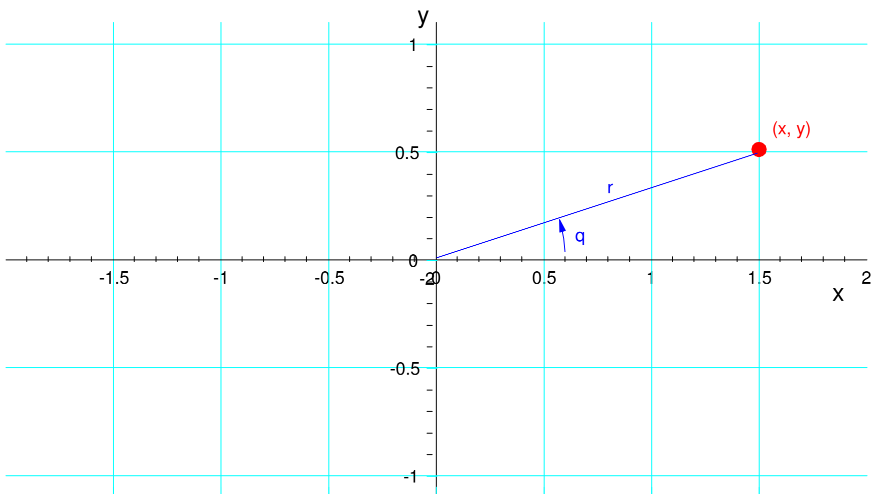
Polar Coordinates

Given any point, you can express it in Cartesian coordinates with its x and y values:

P = (x, y)

You can also express this in polar form as

P = $r\angle\theta$



A point P can be expressed in Cartesian coordinates (x,y) or polar coordinates ($r\Delta\theta$)

The conversion from one to the other is

$$x = r\cos\theta$$

$$y = r\sin\theta$$

or

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(y, x)$$

There are two arctan() functions in Matlab

- atan(y/x) returns the angle from -pi/2 to +pi/2 (-90 degrees to +90 degrees)
- atan2(y, x) returns the angle

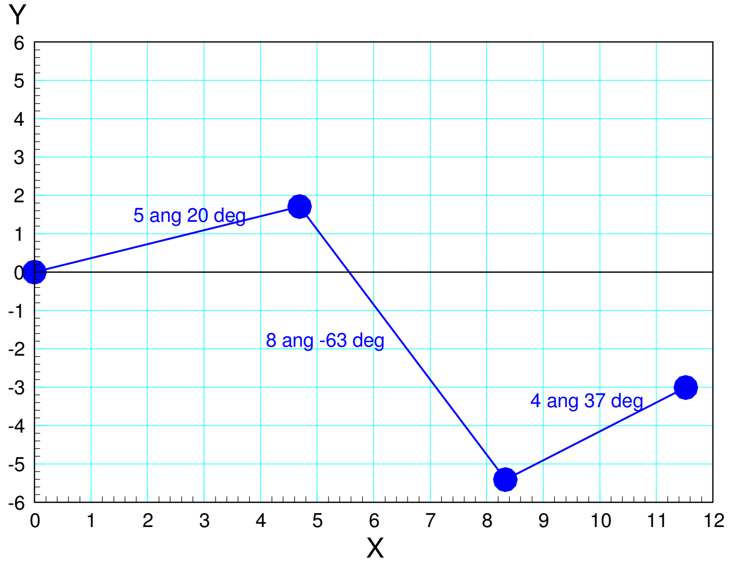
The problem with atan is that if both x and y are negative, the signs cancel. To get the actual angle, you need to use atan2()

Calculations using Polar Coordinates

Using polar-to-rectangular conversions, you can add vectors in polar form. The trick is to convert to rectangular form. Then, the x and y coordinates add.

For example, determine y:

$$y = 5\angle 20^0 + 8\angle -63^0 + 4\angle 37^0$$



To determine y, calculate the x and y coordinate of each term

$$r\angle\theta = (r\cos\theta, r\sin\theta)$$

In Matlab: (note: Matlab uses radians for angles, not degrees)

```
>> x1 = 5*cos(20*pi/180)
x1 =    4.6985

>> y1 = 5*sin(20*pi/180)
y1 =    1.7101

>> x2 = 8*cos(-63*pi/180)
x2 =    3.6319

>> y2 = 8*sin(-63*pi/180)
y2 =   -7.1281

>> x3 = 4*cos(37*pi/180)
x3 =    3.1945

>> y3 = 4*sin(37*pi/180)
y3 =    2.4073
```

The sum is the sum of the x-portion and the sum of the y-portion:

```
>> X = x1+x2+x3
X =   11.5249

>> Y = y1+y2+y3
Y =   -3.0107
```

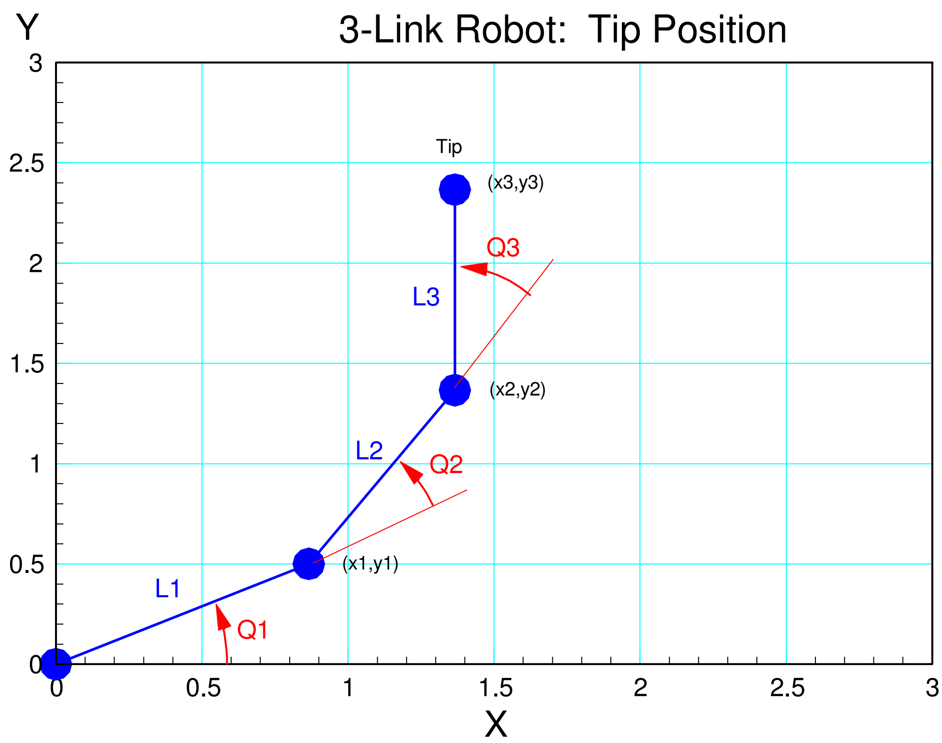
The sum is

$$5\angle 20^0 + 8\angle -63^0 + 4\angle 37^0 = (11.5249, -3.0107)$$

Robotics: Forward Kinematics

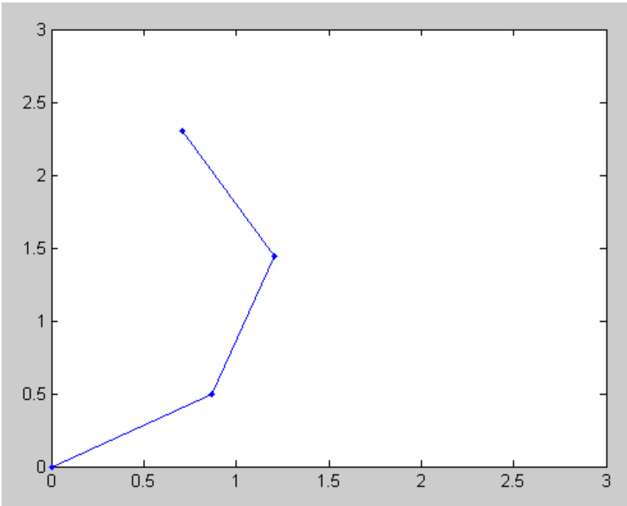
Forward kinematics is the problem of computing where the tip of a robotic arm is given the joint angles. For example, assume

- A 2-dimensional robot
- With three rotational joints, and
- Each link has a length of 1m.



Problem 1: Determine the tip position when the joint angles are {30 degrees, 40 degrees, 50 degrees}

```
function [x3, y3] = ForwardKinematics(q1, q2, q3)

q1 = q1 * pi/180;
q2 = q2 * pi/180;
q3 = q3 * pi/180;

L1 = 1;
L2 = 1;
L3 = 1;

x0 = 0;
y0 = 0;

x1 = L1*cos(q1);
y1 = L1*sin(q1);

x2 = x1 + L2*cos(q1+q2);
y2 = y1 + L2*sin(q1+q2);

x3 = x2 + L3*cos(q1+q2+q3);
y3 = y2 + L3*sin(q1+q2+q3);

plot([x0,x1,x2,x3],[y0,y1,y2,y3],'b.-');
xlim([0,3]);
ylim([0,3]);

pause(0.01);

end


>> [Tx,Ty] = ForwardKinematics(30,40,50)

Tx =    0.7080
Ty =    2.3057
```

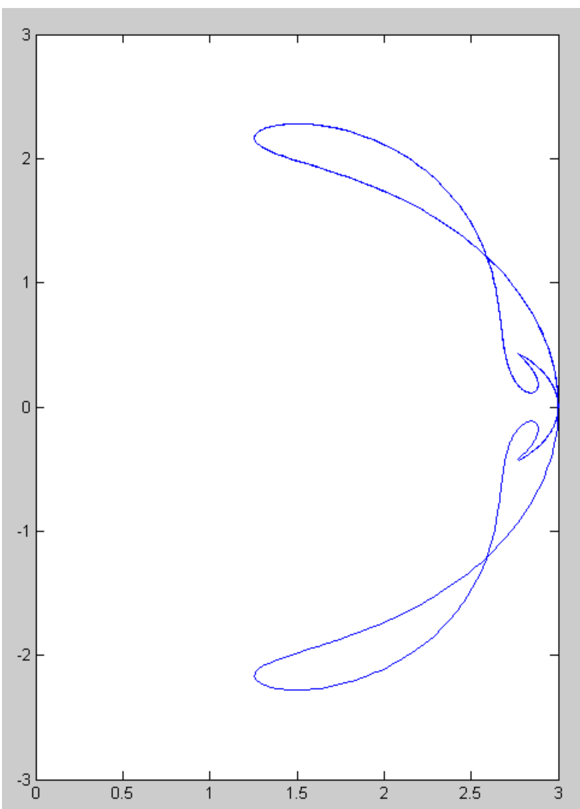


Problem 2: Determine the tip position when the joint angles are

- Q1 = 30*sin(t) degrees
- Q2 = 40*sin(2t) degrees
- Q3 = 50*sin(3t) degrees

Solution:

```
t = [0:0.01:10]';
q1 = 30*sin(t);
q2 = 40*sin(2*t);
q3 = 50*sin(3*t);

Tx = 0*t;
Ty = 0*t;

for i=1:length(t)
    [Tx(i),Ty(i)] = RRR(q1(i), q2(i), q3(i));
    pause(0.01);
end

plot(Tx, Ty)
```



Problem 3: Robot Animation. Show the motion of the robot with the joint angles given in problem #2

```
function [x3, y3] = RRR(q1, q2, q3)

q1 = q1 * pi/180;
q2 = q2 * pi/180;
q3 = q3 * pi/180;

L1 = 1;
L2 = 1;
L3 = 1;

x0 = 0;
y0 = 0;

x1 = L1*cos(q1);
y1 = L1*sin(q1);

x2 = x1 + L2*cos(q1+q2);
y2 = y1 + L2*sin(q1+q2);

x3 = x2 + L3*cos(q1+q2+q3);
y3 = y2 + L3*sin(q1+q2+q3);

plot([x0,x1,x2,x3],[y0,y1,y2,y3],'b.-');
xlim([-3,3]);
ylim([-3,3]);

end
```

Calling Sequence:

Robotics: Inverse Kinematics & fminsearch()

Forward kinematics determines the tip position of a robotic arm given the joint angles. For the three-link arm from before, the solution is

$$x_3 = \cos(\theta_1) + \cos(\theta_1 + \theta_2) + \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_3 = \sin(\theta_1) + \sin(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2 + \theta_3)$$

Inverse kinematics determines the joint angles of a robotic arm given the tip position. This is not that easy to do using algebra. Fortunately, there's a solution in Matlab: the function *fminsearch()*

Fminsearch() is a really useful Matlab command which finds the minimum of a function. For example, find the square root of two.

To use *fminsearch*, first create a function whose minimum is your solution. To find the square root of two, you could use this:

```
function [J] = root2(x)
   e = x*x - 2;
   J = e^2;
   end
```

(save this as root2.m). This function has a minimum when x = $\sqrt{2}$. From the command window, you can guess and guess again to find the solution (the value of x which results in a result of zero):

```
>> root2(3)

ans =    49

>> root2(2)

ans =     4

>> root2(1.4)

ans =    0.0016
```

or you can let Matlab guess for you

```
>> [z,e] = fminsearch('root2',4)

z =    1.4143
e =  1.5665e-008
```

The first line calls function *fmsearch*, telling it to optimize function *root2*, starting with an initial guess of four.

fmsearch then returns the best it could do:

```
z = 1.4143,
```

and the resulting minimum cost if cound determine

```
e = 1.556e-008
```

That's probably not real impressive, so let's look at another example: determine the shape of a hanging chain

- Length = 13 meters
- y(0) = 7
- y(10) = 5

A hanging chain minimizes the potential energy of the chain. Since this is a minimization problem, it's perfect for *fminsearch*.

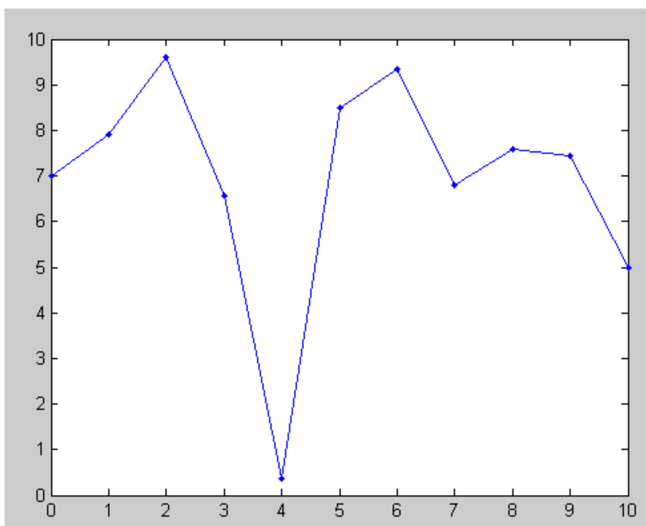
First, write a cost function which

- Is passed your guess for the y-coordinate of the chain from 1 to 9
- Computes the total lenngth of the chain (it should be 13 meters), and
- The total potential energy of the chain

```
function [ J ] = cost_chain( z )
 % [Z,e] = fminsearch('cost_chain', 10*rand(9,1))
 % ECE 111 Lecture #3: fminsearch
 % Shape of a hanging chain that's 13 meters long

   Y = [7,z(1),z(2),z(3),z(4),z(5),z(6),z(7),z(8),z(9),5]';
   PE = sum(Y);
   L = 0;
   for i=2:11
       L = L + sqrt(1 + (Y(i) - Y(i-1))^2);
   end
   E = 13-L;
   J = PE + 100*E*E;
   plot([0:10]', Y, '.-');
   ylim([0,10]);
   pause(0.01);
end
```

Start with an initial guess for the shape of the chain:

```
>> y = 10*rand(9,1);
>> cost_chain(y)

ans =  2.8806e+004
```



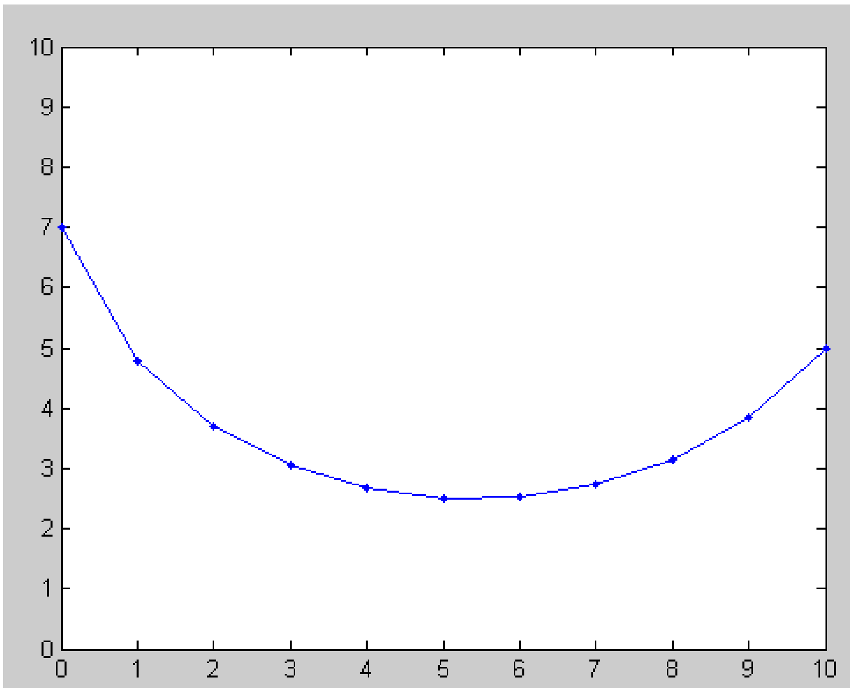
Let *fminsearch* try to optimize this funciton

```
>> [z,e] = fminsearch('cost_chain', y)

Exiting: Maximum number of function evaluations has been exceeded
         - increase MaxFunEvals option.
         Current function value: 41.064042
```

Let *fminsearch* keep going, picking up where you left off:

```
>> [z,e] = fminsearch('cost_chain', z)
```

What you have is a numeric solution to the shape of a hanging chain.



Example 3: Find the joint angles that place a RRR robot at (x=1, y=2).

$$x_3 = 1 = \cos(\theta_1) + \cos(\theta_1 + \theta_2) + \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_3 = 2 = \sin(\theta_1) + \sin(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2 + \theta_3)$$

Solution: Create a function which

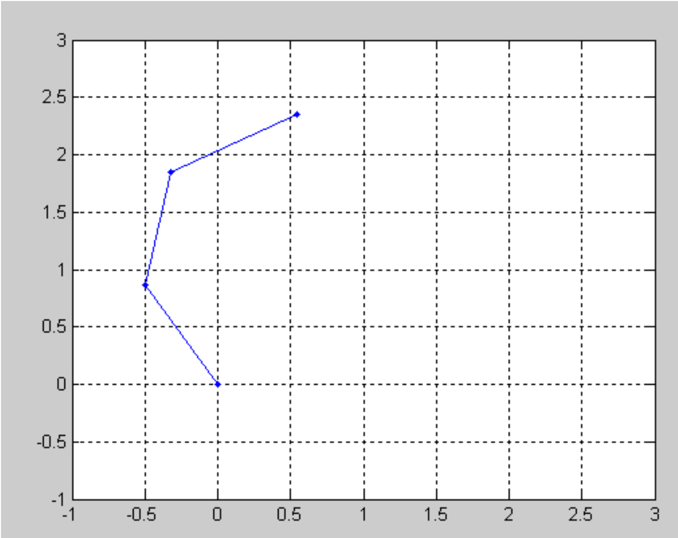
- Is passed the joint angles
- Computes the tip position,
- Computes the error in the tip position, and
- Returns the sum-squared error

```
function [J] = cost_RRR(Q)
```

```
% Tip position
Tx = 1;
Ty = 2;

[x3, y3] = RRR(Q(1), Q(2), Q(3));
pause(0.01);
Ex = x3 - Tx;
Ey = y3 - Ty;

J = Ex^2 + Ey^2;

end
```

Check by calling this function from the command window:

```
>> cost_RRR([120,-40,-50])

ans =     0.3350
```



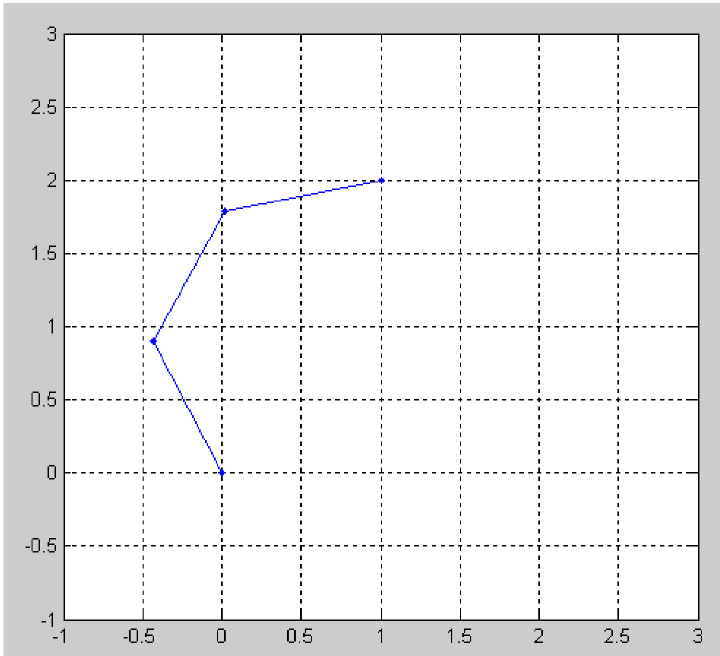
Optimize the function by using fminsearch()

```
>> [Q,e] = fminsearch('cost_RRR',[120,-40,-50])

Q =   115.8522   -53.1532   -50.4906

e =   3.4795e-013
```

One set of joint angles which place the tip at (x = 1, y = 2) is

- q1 = 115.8522 degrees
- q2 = -53.1532 degrees
- q3 = -50.4906 degrees

(there are other solutions)



Summary

- Trig is all about circles
- With sine and cosine functions, you can convert to and from polar coordinates
- With sine and cosine functions, you can compute the tip position of a robotic arm (forward kinematics), and
- With *fmisearch*, you can compute the joint angles which place the tip position of a robot