Complex Numbers

Topics

- Complex Numbers
- Complex Numbers on an HP42 Calculator

Introduction

Numbers matter. Numbers make a difference. The existence of mathematical oddities can determine the fate of empires.

The Number Zero: Take for example the number zero. You might wonder why we have the number zero and if it's needed. Actually, you can get by without the number zero. The Roman Empire, for example, was vast with commerce that stretched from Scotland to Persia - and the Romans did that without the number zero.

Rome was a vast empire with commerce stretching from England to Persia

Romans represented numbers with letters:

- I = 1
- V = 5
- X = 10
- L = 50
- C = 100

The number 27, for example would be represented as XXVII (10 + 10 + 5 + 1 + 1). We know the Romans knew how to do addition, subtraction, multiplication, and division using their number system: you can't run an empire of that scale without such operations. That doesn't mean it was easy. Take, for example, the sum of two numbers:

XXVII + CIX = ?

or the product of two numbers:

XXVII * CIX = ?

In contrast, the Arabic number system uses place holders: each digit represents an increasing power of ten. With this number system, you *need* the number zero. For example 106 apples is very different from 16 apples. You need that zero to represent zero tens as well as to shift the 1 over to the hundreds place.

With Arabic numbers, addition, subtraction become much easier. Multiplication and division remains a challenge, but it is far easier than with Roman numbers.

Negative Numbers: As a second example, consider negative numbers. What does negative one apple mean? Why do we even have negative numbers?

Like the number zero, negative numbers are not necessary. They do make some problems much easier to solve, however.

We've seen that when writing voltage node equations. If you have a circuit, you may or may not know which way the current is flowing by inspection. If you don't mind negative numbers, it doesn't matter. Simply sum the currents from a node to zero and solve. If it turns out a current was flowing the opposite direction of what you assumed, it's not a problem: the current simply shows up as a negative numbers.

Another example comes from the field of accounting. Florence, for example, was a world power in the 13th century. That is a remarkable feat considering that Florence is a small city in Italy, competing against much larger countries such as Spain, France, and England.

One reason for this was the invention of the double-entry bookkeeping system. This allowed merchants in Florence to keep track of expenses (debits, akin to negative numbers) and credits for a given venture. The Florentine merchants would likewise focus on investments where credits exceeded debits (i.e. had a positive profit). That's in contrast to other countries whose merchants focused on how much money came in when a ship entered port, ignoring how much was spent on the whole endeavor.


Republic of Florence: a small city state competing for European dominance (Wikipedia)

Complex Numbers: **T**he first part of Circuits I deals with the DC analysis of resistor circuits. When dealing DC signals, real numbers work just fine:

- Voltages can be expressed by a real number
- Currents can be expressed with real numbers, and
- Resistance's can be expressed with real numbers.

Towards the end of Circuits I and throughout Circuits II, however, we start to work with sinusoidal signals.

The signals sin(t) and cos(t) look like the following:
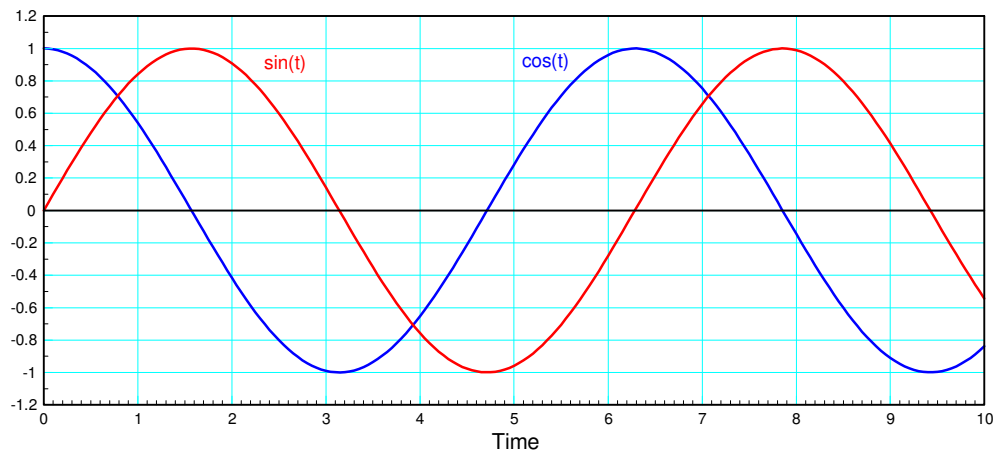
sin(t) and cos(t). Note that the period is 2 pi

These can be combined to create a generalized sinusoid:

$$x(t) = a \cos(\omega t) + b \sin(\omega t)$$

which can also be written as

$$x(t) = r \cos(\omega t + \theta)$$

(more on this later).

What this means is, unlike DC, you need three parameters to define a sinusoid:

- The frequency (w), and
- The cosine and sine terms (a, and b), or
- The amplitude and phase shift (r and θ)

The frequency isn't a problem: if you have a linear circuit, the frequency at all points will be the same as the input. Likewise we typically know what w is. The challenge with circuit analysis with sinusoidal inputs is how to find the other two terms.

This is where complex numbers come into play. If you don't mind using complex numbers, AC analysis of circuits is just like DC analysis - only your answers will be complex numbers. Likewise, all ECE courses past Circuits I rely upon using complex numbers.

In short, ECE majors use complex numbers - probably more than any other major. Fortunately, Matlab (and HP calculators) have no problem working with complex numbers.

Complex Numbers

Unlike real numbers, complex numbers have two terms. This allows us to represent the cosine and sine terms for a sinusoid with a single (albeit complex) number.

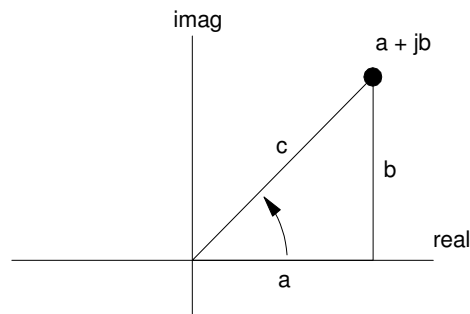The basic idea behind complex numbers is to define a term, j, as[1]

$$j \equiv \sqrt{-1}$$

Any given number can then have a real and a complex part

$$x = a + jb$$

You can express this number in rectangular form (a + jb) or polar form

$$x = c \angle \theta$$

Complex Number (a + jb) can also be expressed as $(c \angle \theta)$

Note: The polar form is shorthand notation and actually means

$$c \angle \theta \equiv c \cdot e^{j\theta}$$

The complex exponential has two terms

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

This leads to Euler's identity

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

---

[1] Math majors call this term i for imaginary. In ECE, i means current, so we use the letter j to represent the complex part of a number.

Proof: Substitute for the complex exponential

$$\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right) = \tfrac{1}{2}((\cos\theta + j\sin\theta) + (\cos(-\theta) + j\sin(-\theta)))$$

Using some trig identities

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

gives

$$= \left(\tfrac{1}{2}\right)(\cos\theta + j\sin\theta + \cos\theta - j\sin\theta)$$

$$= \tfrac{1}{2}(2\cos\theta)$$

$$= \cos\theta$$

For the sine term:

$$\left(\frac{e^{j\theta}-e^{-j\theta}}{2j}\right) = \left(\tfrac{1}{2j}\right)((\cos\theta + j\sin\theta) - (\cos(-\theta) + j\sin(-\theta)))$$

$$= \left(\tfrac{1}{2j}\right)(\cos\theta + j\sin\theta - \cos\theta - (-\sin\theta))$$

$$= \left(\tfrac{1}{2j}\right)(j\sin\theta + j\sin\theta)$$

$$= \sin\theta$$

Inputting Complex Numbers into Matlab

With Matlab, the default value of j is $\sqrt{-1}$. If you redefine j, this no longer holds, but you can restore this as

```
>> j = sqrt(-1)
```

To input a number into Matlab in rectangular form, simply use the j variable

```
>> A = 2 + j*3

A =    2.0000 + 3.0000i
```

You can also input a variable in polar form. $B = 3\angle 1.5$ is input as

```
>> B = 3 * exp(j*1.5)

B =    0.2122 + 2.9925i
```

(note: Matlab uses radians for its angle units).

The default display in Matlab is rectangular units. To convert to polar, use the abs and angle

```
>> abs(B)

ans =     3

>> angle(B)

ans =    1.5000
```

Inputting Complex Numbers into an HP42

This uses the complex key. In rectangular mode (yellow - MODES - RECT)
- the y register becomes the real part
- the x register becomes the complex part.

To input the number 2 + j3, press

```
MODES
RECT
2
enter
3
complex
```

In polar mode (yellow - MODES - POLAR)
- the y register becomes the magnitude
- the x register becomes the angle (in the current units degrees / rad / grad)

To input the number $5\angle 27^0$
```
MODES
POLAR
5
enter
27
complex
```

Addition, Subtraction, Multiplication, and Division

With complex numbers, you can add, subtract, multiply, and divide just like real numbers:

Addition: For addition
- The real parts add, and
- The complex parts add.

Example:

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction: Again, the real parts subtract and the complex parts subtract

$$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$$

Note: Addition and subtraction also work in polar form. This requires a polar to rectangular conversion:

$$r_1 \angle \theta_1 + r_2 \angle \theta_2 = (r_1 \cos \theta_1) + j(r_1 \sin \theta_1) + (r_2 \cos \theta_2) + j(r_2 \sin \theta_2)$$
$$= (r_1 \cos \theta_1 + r_2 \cos \theta_2) + j(r_1 \sin \theta_1 + r_2 \sin \theta_2)$$

Moral: Addition and subtraction is easier in rectangular form. Or use a calculator that can add and subtract complex numbers.

Multiplication: Multiplication is a little trickier, but the result is a complex number.

$$(a_1 + jb_1)(a_2 + jb_2) = a_1 a_2 + j a_1 b_2 + j a_2 b_1 + j^2 b_1 b_2$$

Note that j^2 = -1

$$(a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

Polar form actually works better for multiplication

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2}$$
$$= r_1 r_2 e^{j\theta_1} e^{j\theta_2}$$

Using the property

$$e^a e^b = e^{a+b}$$

gives

$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 e^{j\theta_1 + j\theta_2}$$
$$r_1 \angle \theta_1 \cdot r_2 \angle \theta_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

When you multiply complex numbers
- The magnitude multiplies and
- The angles add

Complex Conjugate: The complex conjugate of a complex number is

$$(a + jb)^* \equiv a - jb$$

The complex conjugate has the property that the product of a complex number with its complex conjugate is a real number, equal to the magnitude squared:

$$(a + jb) \cdot (a - jb) = a^2 + b^2$$

Division: Division also results in a complex number but takes even more computations. It uses the complex conjugate of the denominator:

$$\left(\frac{a_1 + jb_1}{a_2 + jb_2} \right) = \left(\frac{a_1 + jb_1}{a_2 + jb_2} \right) \left(\frac{a_2 - jb_2}{a_2 - jb_2} \right)$$

$$= \left(\frac{(a_1 a_2 + b_1 b_2) + j(-a_1 b_2 + a_2 b_1)}{a_2^2 + b_2^2} \right)$$

$$= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + j \left(\frac{-a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2} \right)$$

Polar form is again simpler for division

$$\left(\frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} \right) = \left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \right)$$

$$= \left(\frac{r_1}{r_2} \right) e^{j\theta_1} e^{-j\theta_2}$$

$$= \left(\frac{r_1}{r_2} \right) e^{j\theta_1 - k\theta_2}$$

$$= \left(\frac{r_1}{r_2} \right) \angle (\theta_1 - \theta_2)$$

The division of complex numbers is

- The ratio of the magnitudes and
- The difference in the angles.

Sample Problems

Problem 1: Find y

$$y = \left(\frac{(2+j3)(4+j5)+(6+j7)}{8+j9} \right)$$

By Hand: (pretty painful)

$$= \left(\frac{((8-15)+j(12+10))+(6+j7)}{(8+j9)} \right)$$

$$= \left(\frac{(-7+j22)+(6+j7)}{(8+j9)} \right)$$

$$= \left(\frac{-1+j29}{8+j9} \right)\left(\frac{8-j9}{8-j9} \right)$$

$$= \left(\frac{(-8+261)+j(232+9)}{64+81} \right)$$

$$= \left(\frac{253+j241}{145} \right)$$

$$= \left(\frac{253}{145} \right) + j\left(\frac{241}{145} \right)$$

$$1.7448 + j1.6621$$

Using Matlab

```
>> y = ( (2+j*3)*(4+j*5) + (6+j*7) ) / (8 + j*9)

y =   1.7448 + 1.6621i
```

Using an HP42

```
2
enter
3
complex
4
enter
6
complex
*
6
enter
7
complex
+
8
enter
9
complex
/
ans = 1.7448 + j1.6621
```

Partial Fraction Expansion with Real Poles

A common problem in ECE is to expand a function by its roots. For example, find

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)} \right) = \left(\frac{a}{x+1} \right) + \left(\frac{b}{x+2} \right) + \left(\frac{c}{x+3} \right)$$

Solution #1: (the hard way) Place the right side over a common denominator and match coefficients.

$$= \left(\frac{a}{x+1} \right)\left(\frac{(x+2)(x+3)}{(x+2)(x+3)} \right) + \left(\frac{b}{x+2} \right)\left(\frac{(x+1)(x+3)}{(x+1)(x+3)} \right) + \left(\frac{c}{x+3} \right)\left(\frac{(x+1)(x+2)}{(x+1)(x+2)} \right)$$

This places all terms over a common denominator. The numerator is then

$$2x + 3 = a(x+2)(x+3) + b(x+1)(x+3) + c(x+1)(x+2)$$

$$2x + 3 = a(x^2 + 5x + 6) + b(x^2 + 4x + 3) + c(x^2 + 3x + 2)$$

This gives three equations for three unknowns.

Matching the x^2 terms:

$$0 = a + b + c$$

x^1 terms:

$$2 = 5a + 4b + 3c$$

x^0 terms:

$$3 = 6a + 3b + 2c$$

Place in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Solve in Matlab

```
>> A = [1,1,1;5,4,3;6,3,2]

     1     1     1
     5     4     3
     6     3     2

>> B = [0;2;3]

     0
     2
     3
```

```
>> inv(A)*B

   a    0.5000
   b    1.0000
   c   -1.5000
```

so

$$\left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \left(\frac{0.5}{x+1}\right) + \left(\frac{1}{x+2}\right) + \left(\frac{-1.5}{x+3}\right)$$

Solution #2: (cover-up method). Equals is a powerful symbol: it means both sides are identical everywhere.

The right side blows up (goes to infinity) near x = {-1, -2, -3}. The left side has to match.

Near x = -1, only the first term matters since it's going to infinity while the other terms are finite. So

$$\lim_{x \to -1} \left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \lim_{x \to -1} \left(\frac{a}{x+1}\right)$$

Cancel (cover up) the (x+1) term and evaluate

$$a = \left(\frac{2x+3}{(x+2)(x+3)}\right)_{x=-1} = 0.5$$

Similarly, near x = -2, only the second term (b) matters on the right.

$$\lim_{x \to -2} \left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \lim_{x \to -2} \left(\frac{b}{x+2}\right)$$

Cancel (cover up) the (x+1) term and evaluate

$$b = \left(\frac{2x+3}{(x+1)(x+3)}\right)_{x=-2} = 1$$

Near x = -3, only the third term (c) matters:

$$\lim_{x \to -3} \left(\frac{2x+3}{(x+1)(x+2)(x+3)}\right) = \lim_{x \to -3} \left(\frac{c}{x+3}\right)$$

$$c = \left(\frac{2x+3}{(x+1)(x+2)}\right)_{x=-3} = -1.5$$

Either method works - the cover-up method is a lot easier. The cover-up method also works in Matlab. For the limit, take a number close to the point you're evaluating (perturb by 1e-9). Solve for {a, b, c}

```
>> x = -1 + 1e-9;
>> a = (2*x + 3)/( (x+1)*(x+2)*(x+3) )  *  (x+1)

a =     0.5000

>> x = -2 + 1e-9;
>> b = (2*x + 3)/( (x+1)*(x+2)*(x+3) )  *  (x+2)

b =     1.0000

>> x = -3 + 1e-9;
>> c = (2*x + 3)/( (x+1)*(x+2)*(x+3) )  *  (x+3)

c =    -1.5000
```

Partial Fraction with Complex Numbers

Placing all terms over a common denominator works, but is that much harder with complex numbers. The cover-up method is the same either way.

Example: Determine {a, b, c}

$$\left(\frac{5x+7}{(x+1+j3)(x+1-j3)(x+5)} \right) = \left(\frac{a}{x+1+j3} \right) + \left(\frac{b}{x+1-j3} \right) + \left(\frac{c}{x+5} \right)$$

Solving

$$a = \left(\frac{5x+7}{(x+1-j3)(x+5)} \right)_{x=-1-j3} = 0.3600 + j0.3533$$

$$b = \left(\frac{5x+7}{(x+1+j3)(x+5)} \right)_{x=-1+j3} = 0.3600 - j0.3533$$

$$c = \left(\frac{5x+7}{(x+1+j3)(x+1-j3)} \right)_{x=-5} = -0.7200$$

Solving using Matlab

```
>> x = -1 - j*3 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) )  *  (x+1+j*3)

a =    0.3600 + 0.3533i

>> x = -1 + j*3 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) )  *  (x+1-j*3)

a =    0.3600 - 0.3533i

>> x = -5 + 1e-9;
>> a = (5*x+7) / ( (x+1+j*3)*(x+1-j*3)*(x+5) )  *  (x+5)

a =    -0.7200


>>
```

More Fun with Complex Numbers

Note: These use the properties

$$\ln(e^x) = x$$

$$e^{\ln(x)} = x$$

$$(e^a)^b = e^{ab}$$

Example 1: Find y:

$$y = (2 + j3)^{(4+j5)}$$

Solution: Convert to polar form (using radians)

$$2 + j3 = 3.6056\angle 0.9828$$

$$= e^{\ln(3.6056)} \cdot e^{j0.9828}$$

$$= e^{1.2825 + j0.9828}$$

Raise to the power

$$= (e^{1.2825 + j0.9828})^{(4+j5)}$$

$$= e^{(0.2159 + j10.3453)}$$

Separate

$$= e^{0.2159} e^{j10.3435}$$

$$= 1.2410 \cdot (\cos(10.3425) + j\sin(10.3435))$$

$$= 1.2410 \cdot (-0.6068 - j0.7942)$$

$$= -0.7530 - j0.9864$$

Check in Matlab

```
>> y = (2 + j*3) ^ (4 + j*5)

y =   -0.7530 - 0.9864i
```

Note: Matlab was *way* easier.

Check on an HP42:

```
2
enter
3
complex
4
enter
5
complex
y^x

ans = -0.7530 - j0.9864
```

Example 2: Find y:

$$y = \cos(2 + j3)$$

Use Euler's identity

$$\cos(2 + j3) = \left(\tfrac{1}{2}\right)(e^{(2+j3)} + e^{j(2+j3)})$$

$$= \left(\tfrac{1}{2}\right)((e^2 e^{j3}) + (e^{j2} e^{-3}))$$

$$= \left(\tfrac{1}{2}\right)(e^2 \cdot (\cos(3) + j\sin(3)) + e^{-3} \cdot (\cos(2) + j\sin(2))$$

$$= \left(\frac{e^2\cos(3) + e^{-3}\cos(2)}{2}\right) + j\left(\frac{e^2\sin(3) + e^{-3}\sin(2)}{2}\right)$$

Check in Matlab

```
>> cos(2 + j*3)

ans =   -4.1896 - 9.1092i
```

Check on an HP42

```
2
enter
3
complex
COS

ans = -4.1896 - j9.1092
```

Find y:

$$y = \ln(2 + j3)$$

Express as an exponential

$$2 + j3 = 3.6056\angle 0.9828$$

$$= e^{\ln(3.6056)+j0.9828}$$

$$\ln(2 + j3) = \ln(3.6056) + j0.9828$$

Check in Matlab

```
>> log(2 + j*3)

ans =   1.2825 + 0.9828i
```

Check with an HP42

```
2
enter
3
ln

ans =   1.2825 + 0.9828i
```

Moral #1: Pretty much anything you do with real numbers you can do with complex numbers. The answer will be complex though.

Moral #2: When dealing with complex numbers, it is a *lot* easier to use Matlab or an HP calculator than doing it by hand...

Summary:
- Complex numbers allow you to represent something with two degrees of freedom
- Addition, subtraction, multiplication, and division all work with complex numbers
- Evaluating equations with complex numbers by hand is really painful.
- Matlab and HP calculators don't mind complex numbers: they work with them just as easily as they do with real numbers.