Math 165: Calculus I

Differentiation

ECE 111 Introduction to ECE

Jake Glower - Week #6

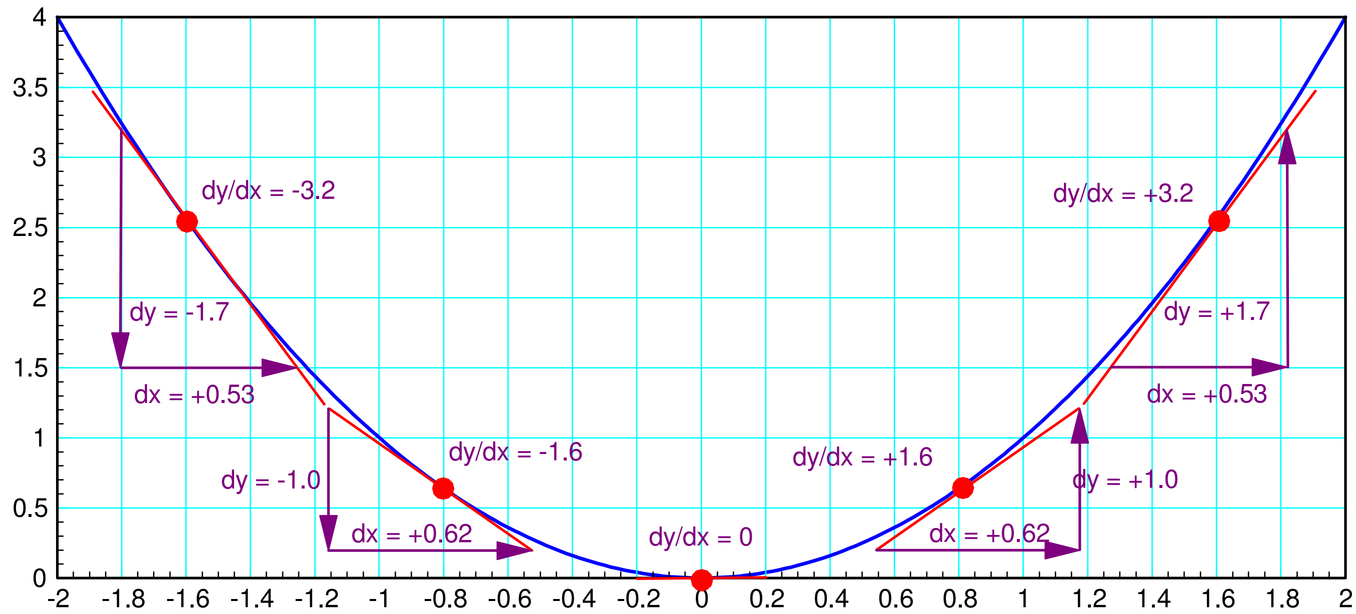
Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Differentiation

The derivative of a function gives you the slope.

This has several uses.

- Can be used to find what's happening at a given time
- Can be used to find the maximum & minimum of a function



Example: Bank Account

The derivative is the slope: this tells you what is happening with your bank account:

0<x<2: The slope is zero

- Nothing is being added or taken out

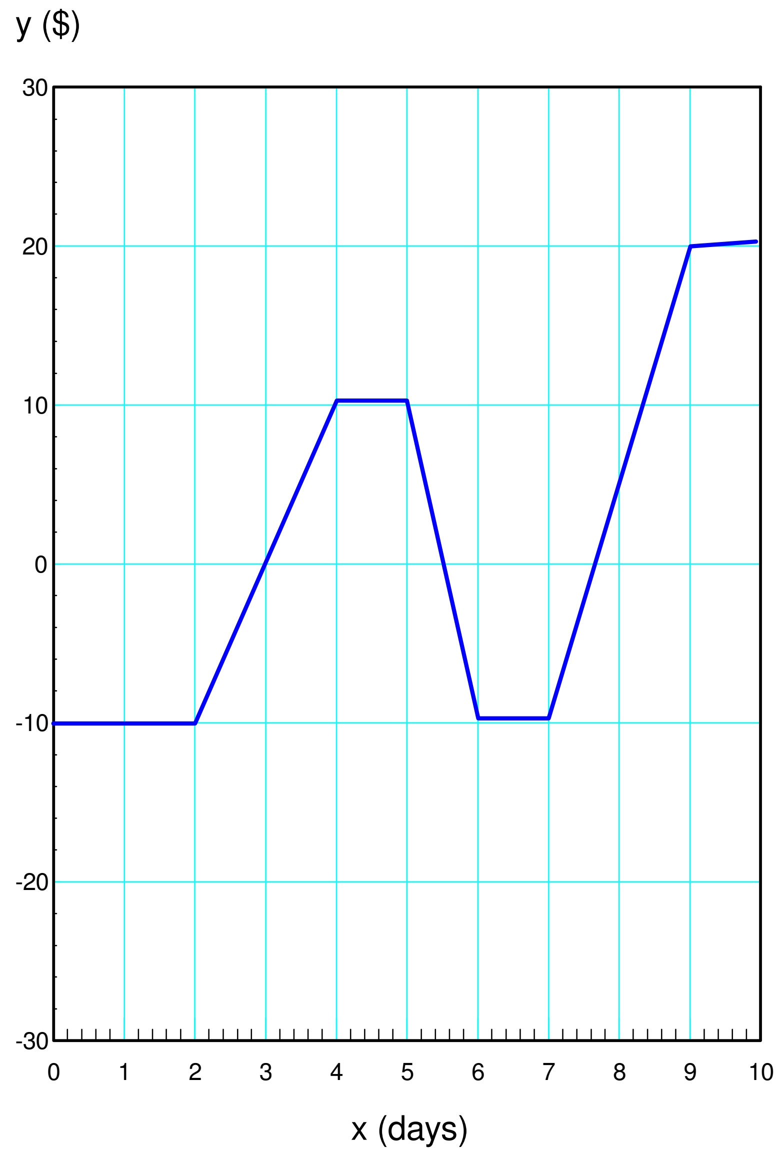
2<x<4: The slope is positive

- Money is being added to your account

5<x<6: Negative slope

- Money is being withdrawn

Furthermore, the derivative also tells you how much money you're depositing or withdrawing.



Calculating the Derivative

Determine the slope:

$$slope = y'(x) = \frac{dy}{dx} = \frac{\text{change in y}}{\text{change in x}}$$

0<x<2

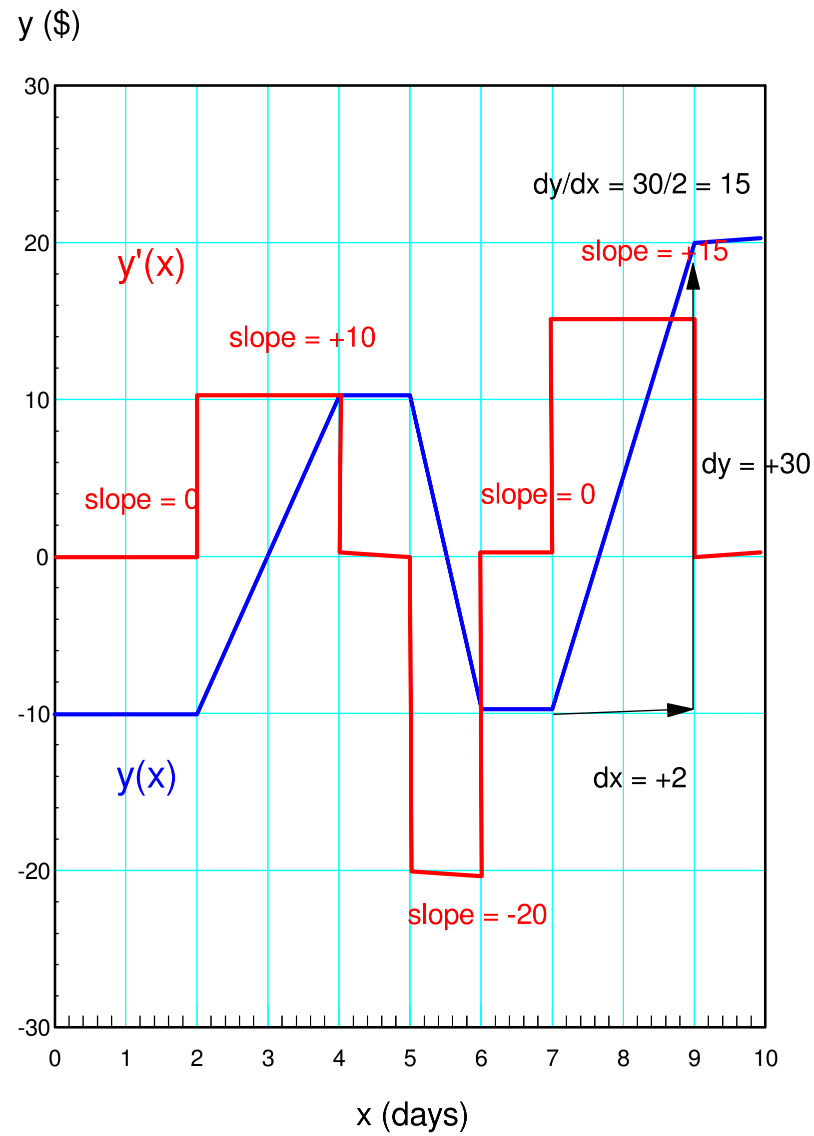
- dy = 0,   dx = 2,   dy/dx = 0

2<x<4

- dy =20, dx= 2, dy/dx = 10

7<x<9

- dy = +30, dx = 2, dy/dx = 15



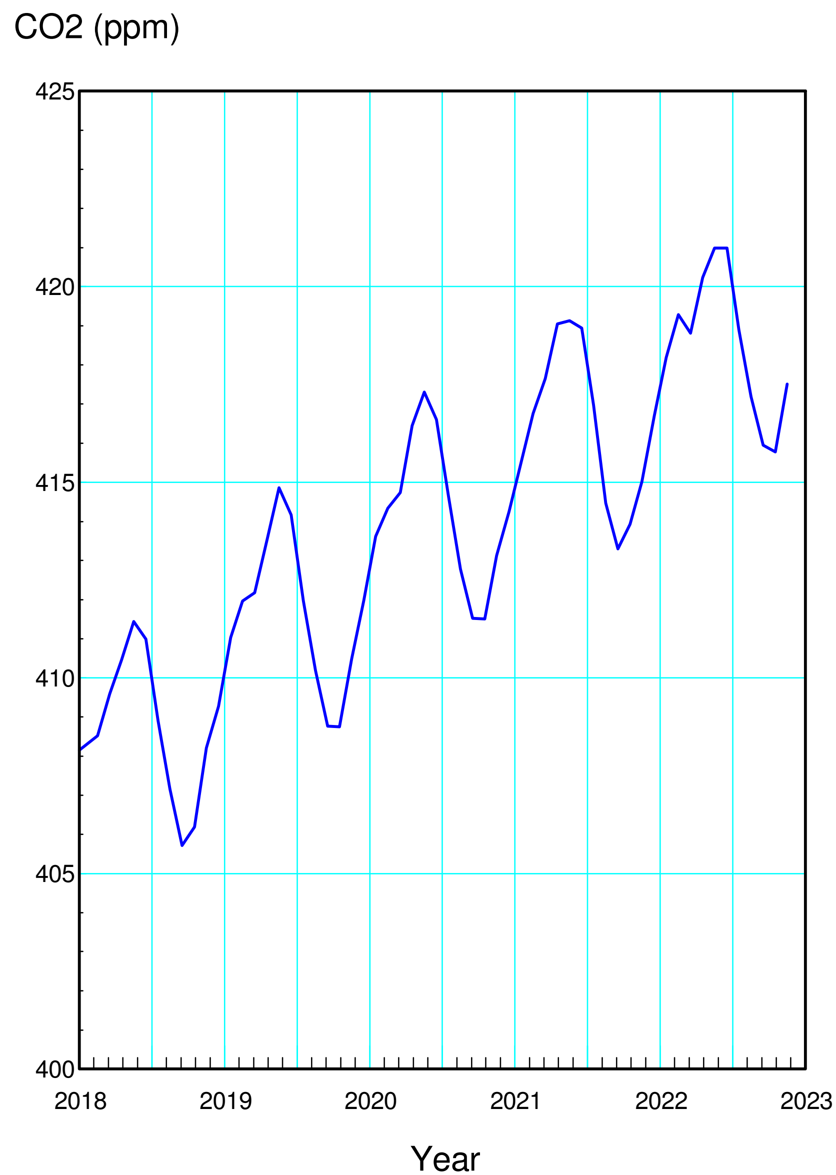
Example: Global CO2 Levels

- 2015 - 2023

Graph shows global CO2 levels

Derivative shows amount of CO2 added

- Positive Slope: CO2 is being added
- Negative Slope: CO2 is bring removed



What's Happening?

Derivatives are useful

- They provide information about what's happening on a month-by-month basis.

Most land is in the northern hemisphere

Summer (July)

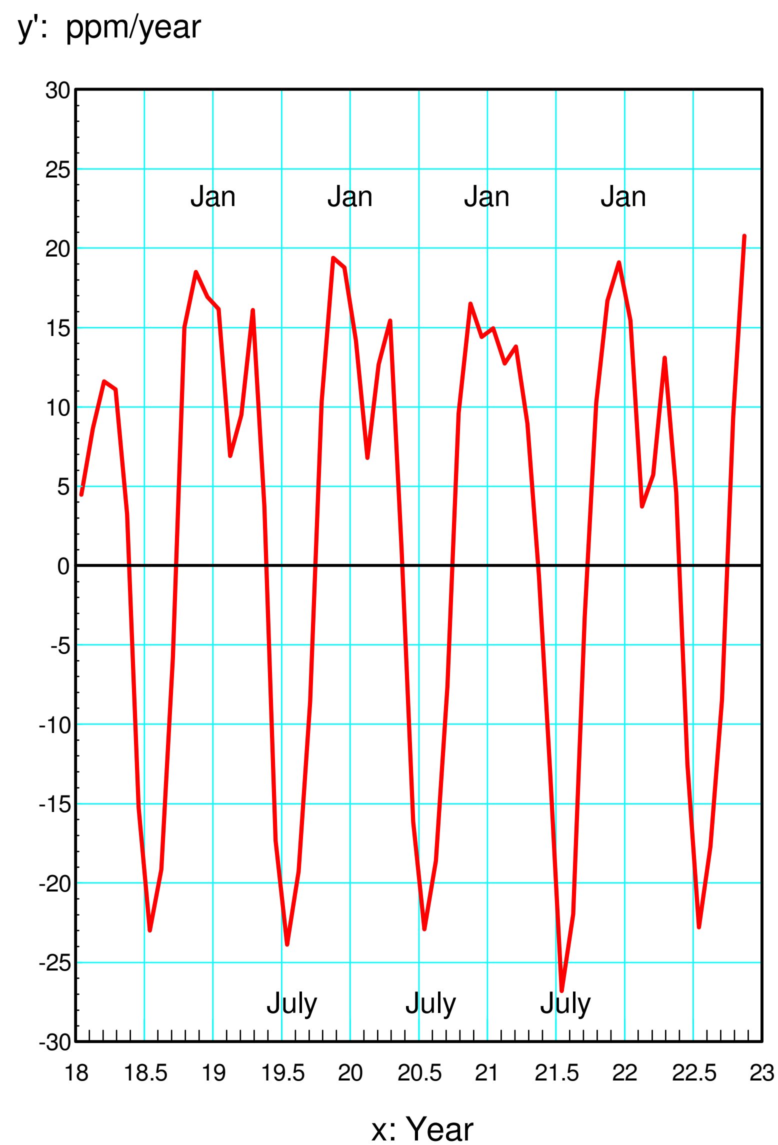
- Global CO_2 levels drop.
- Trees green up and absorb CO_2

Winter (January)

- Global CO_2 levels drop
- Trees drop their leaves
- Leaves decay, releasing CO_2

Also shows a hiccup in March

- Not sure why...



Topics for This Lecture

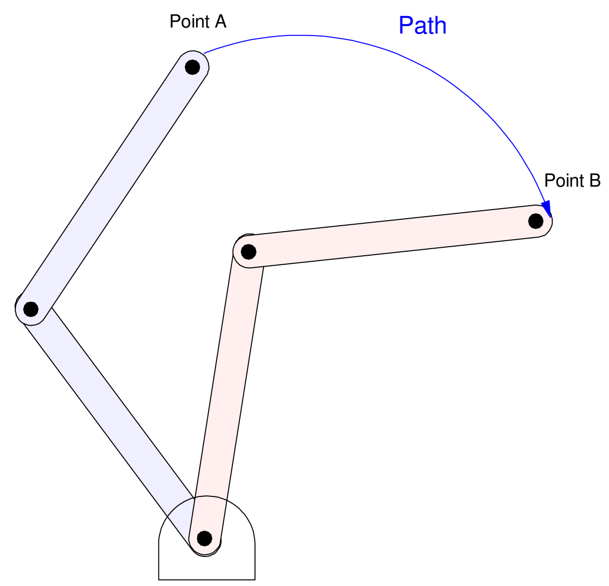
How to find a derivative

- Using lookup tables similar to what you'll do in Math 165,
- Using graphical methods, and
- Using numerical methods (i.e. Matlab).

Evaluate different paths for a robotic arm

- 1st derivative of angle = velocity
  - Velocity = voltage for a DC motor
- 2nd derivative of angle = acceleration
  - Acceleration = current for a DC motor

Different paths result in different implied voltages and currents



Differentiation: Method #1

Memorize and apply a set of rules.

- What you do in Calculus I

Power Functions: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

Exponential $\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$

Sine Function $\frac{d}{dx}(\sin(ax)) = a \cdot \cos(ax)$

Cosine Function $\frac{d}{dx}(\cos(ax)) = -a \cdot \sin(ax)$

Chain Rule $\frac{d}{dx}(f(x) \cdot g(x)) = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$

Division $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{(g(x))^2}$

With these rules, you can find

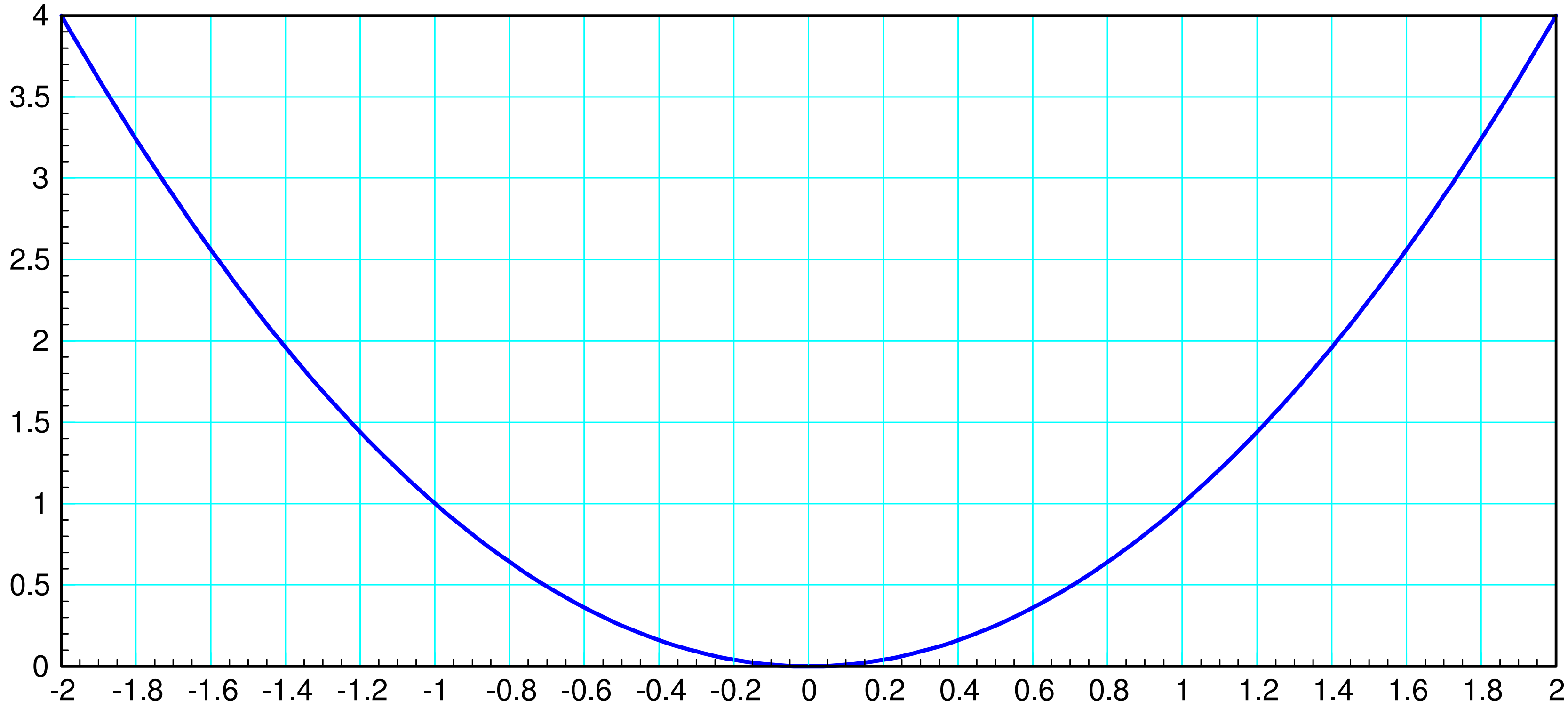
$$y = \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right)$$

Stay tuned until you take Calculus I to solve using this method.

Differentiation Method #2: Graphical

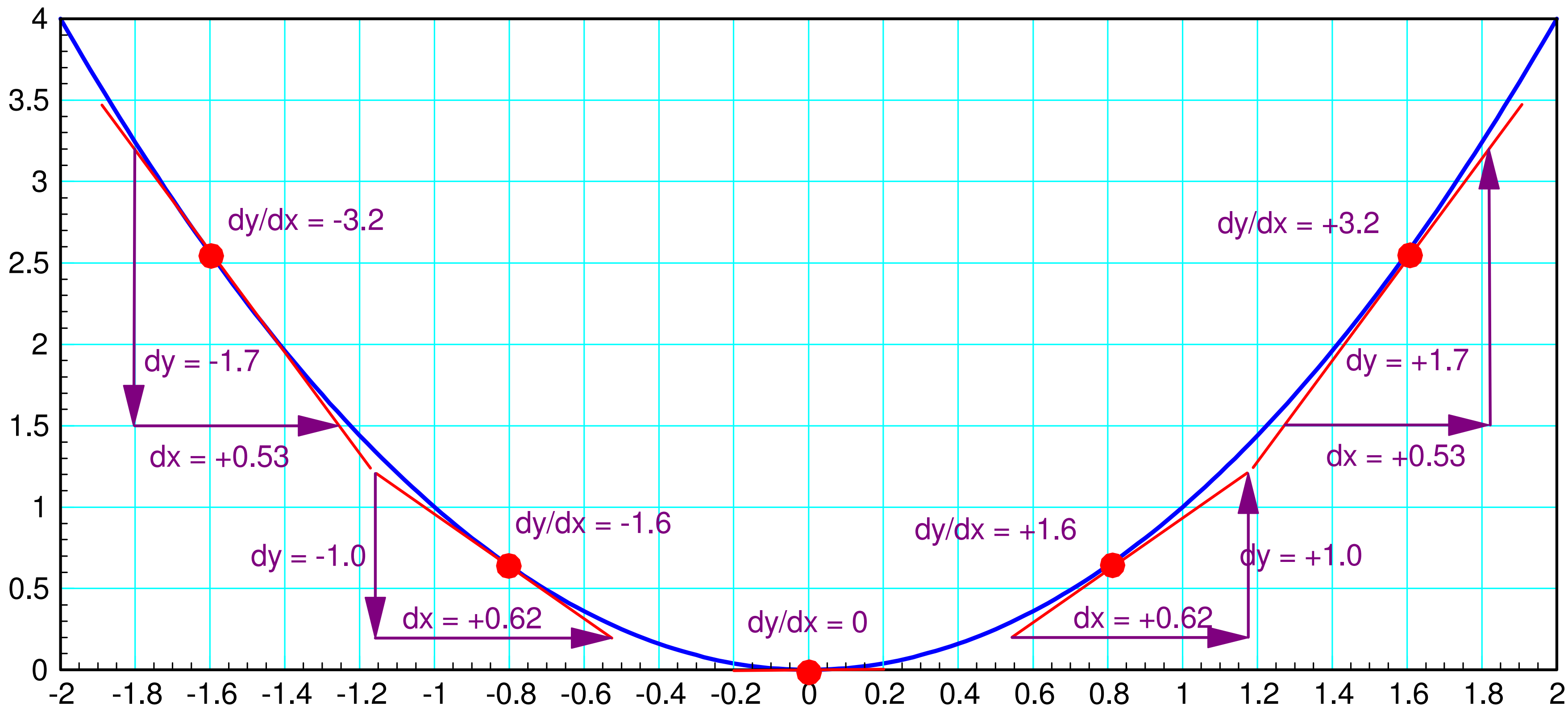
- Draw the tangent at several points
- Determine the slope at each point
- Draw a curve connecting the slopes

Example: Find the derivative of $y = x^2$

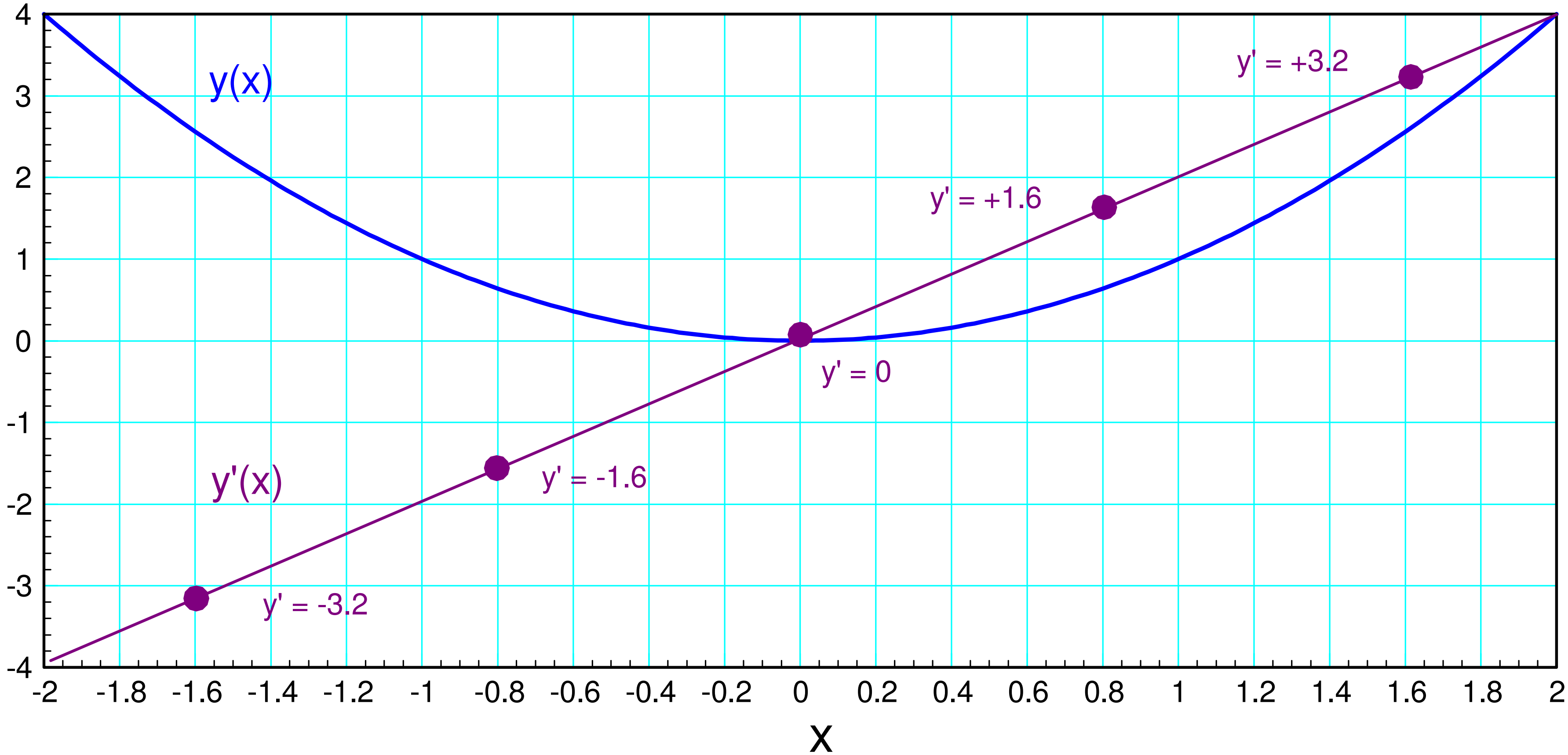


Step 1: Draw the tangent at several points (shown in red)

Step 2: Determine the slope at each point (shown in purple)



Step 3: Connect the points (shown in purple)

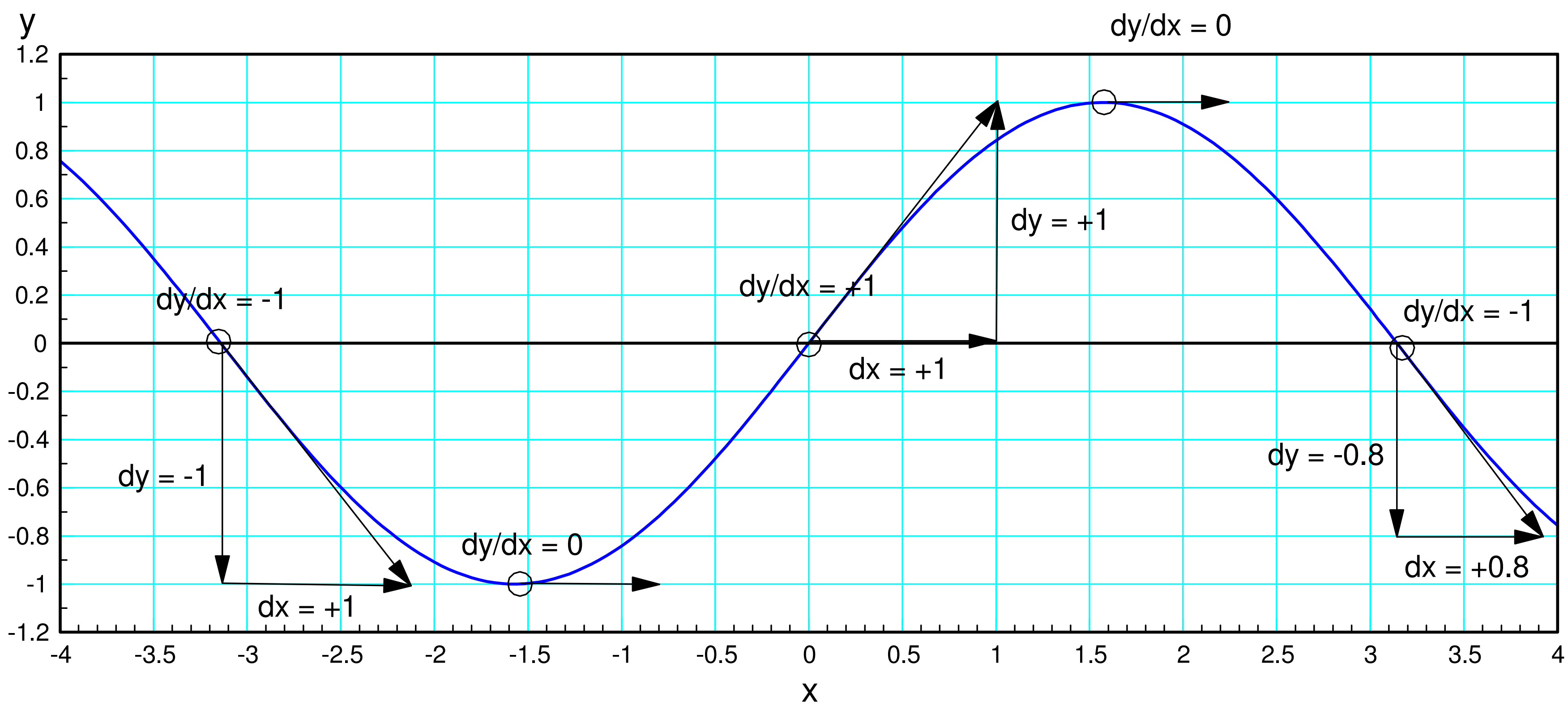


Example #2: Determine the derivative of a sine wave:

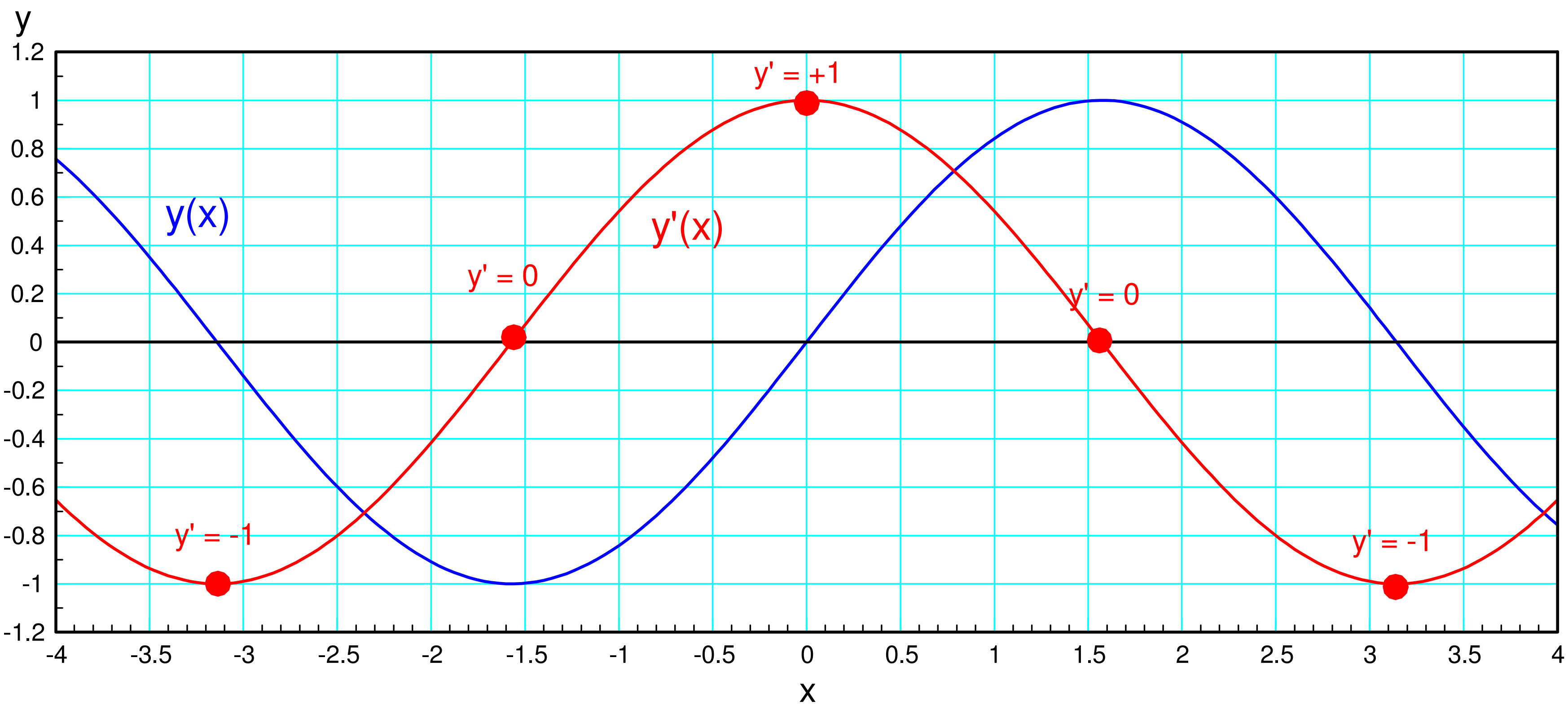
$$y = \sin(x)$$

Step 1: Plot y vs. x (shown below)

Step 2: Compute the slope at several points



Step 3: Plot the slopes and connect them with a smooth curve

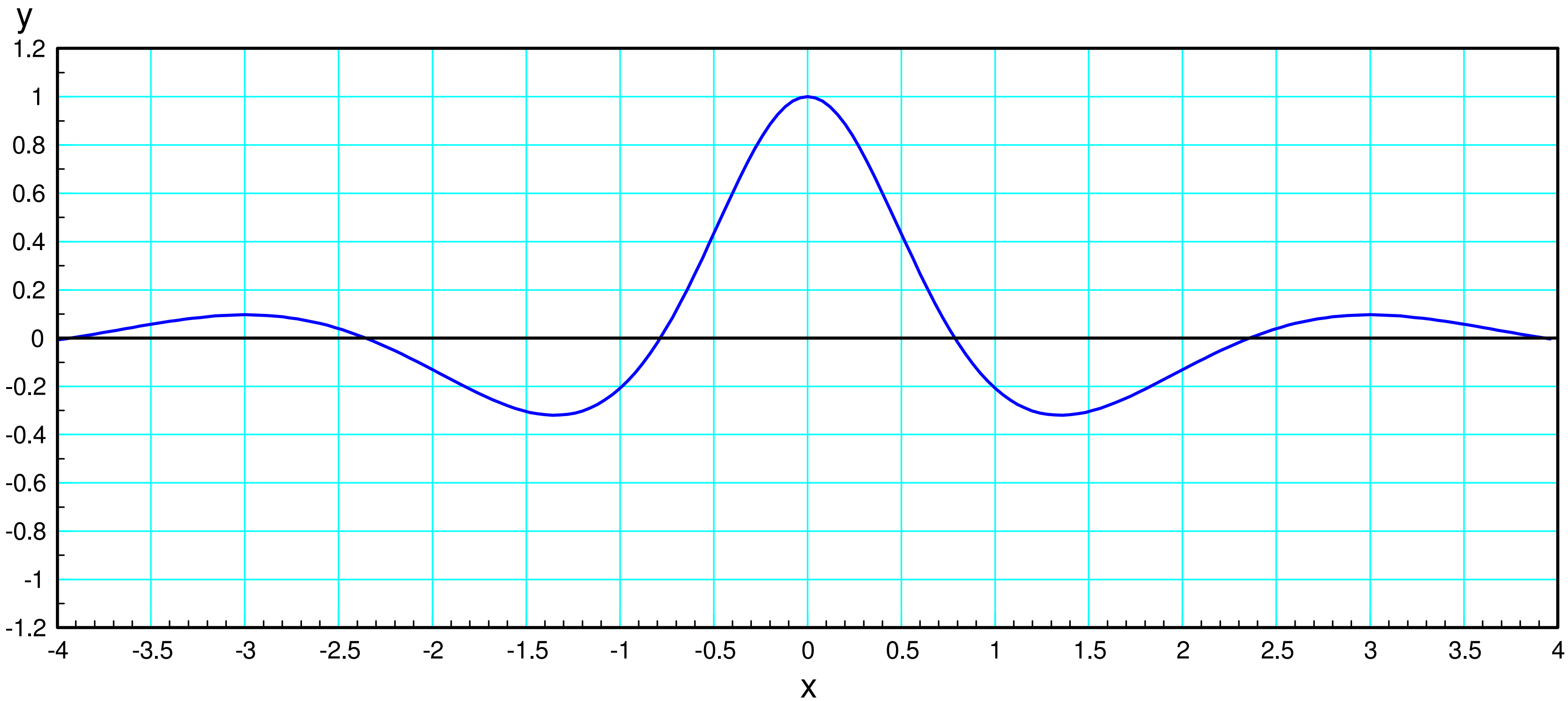


The result (shown in red) is a cosine function

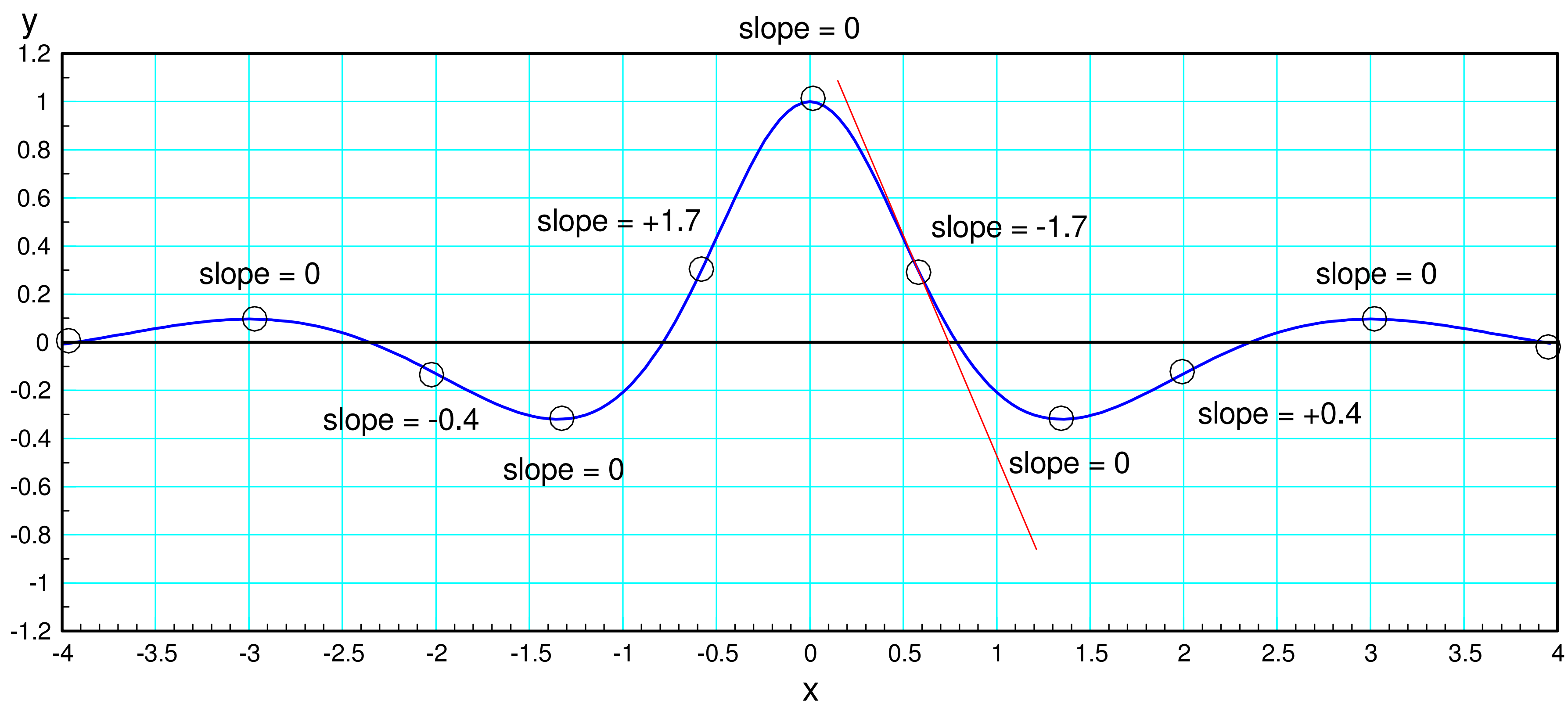
Graphical Example #3:

- Works for any function

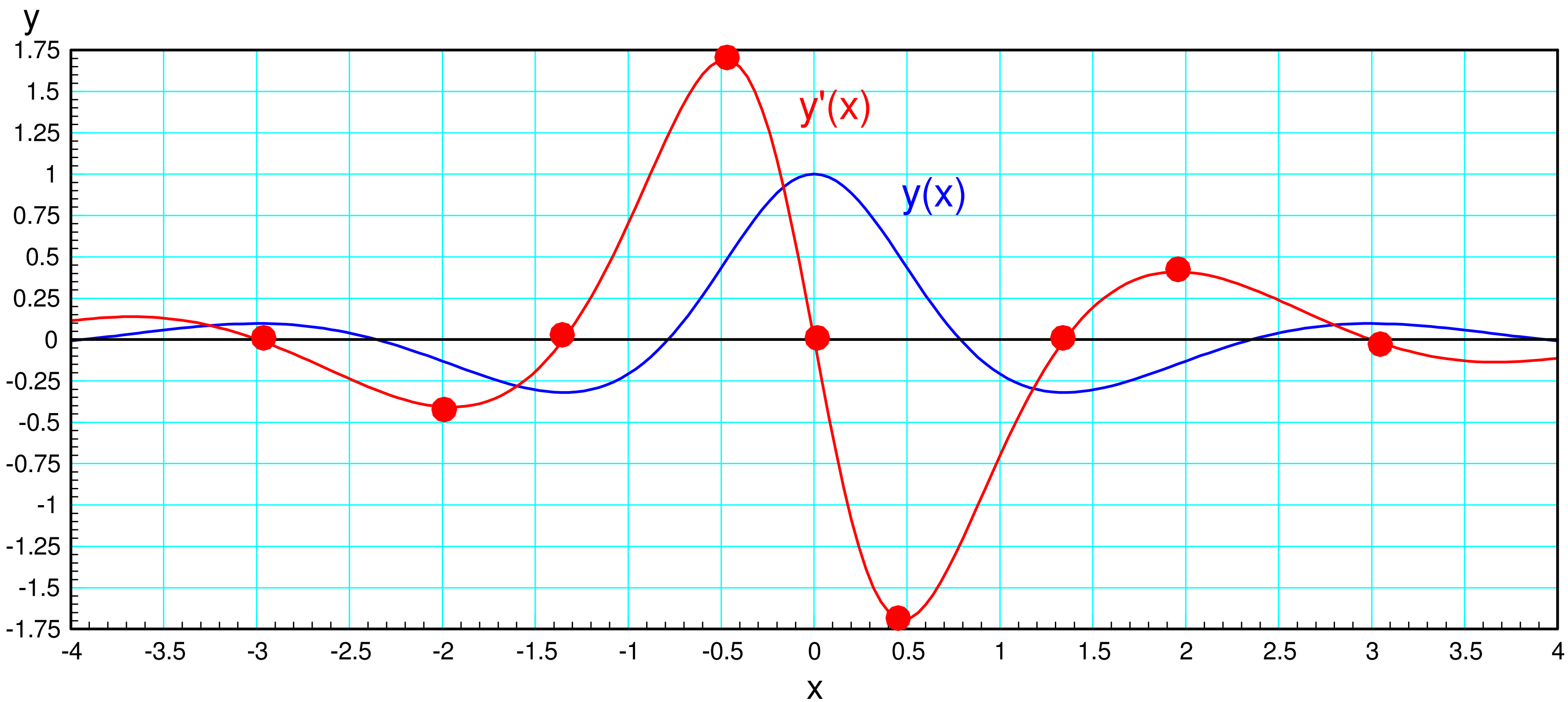
Example: $y = \frac{d}{dx}\left(\frac{\cos(x)}{x^2+1}\right)$



Step 1: Compute the slope at several points



Step 2: Plot the derivatives and connect with a smooth curve



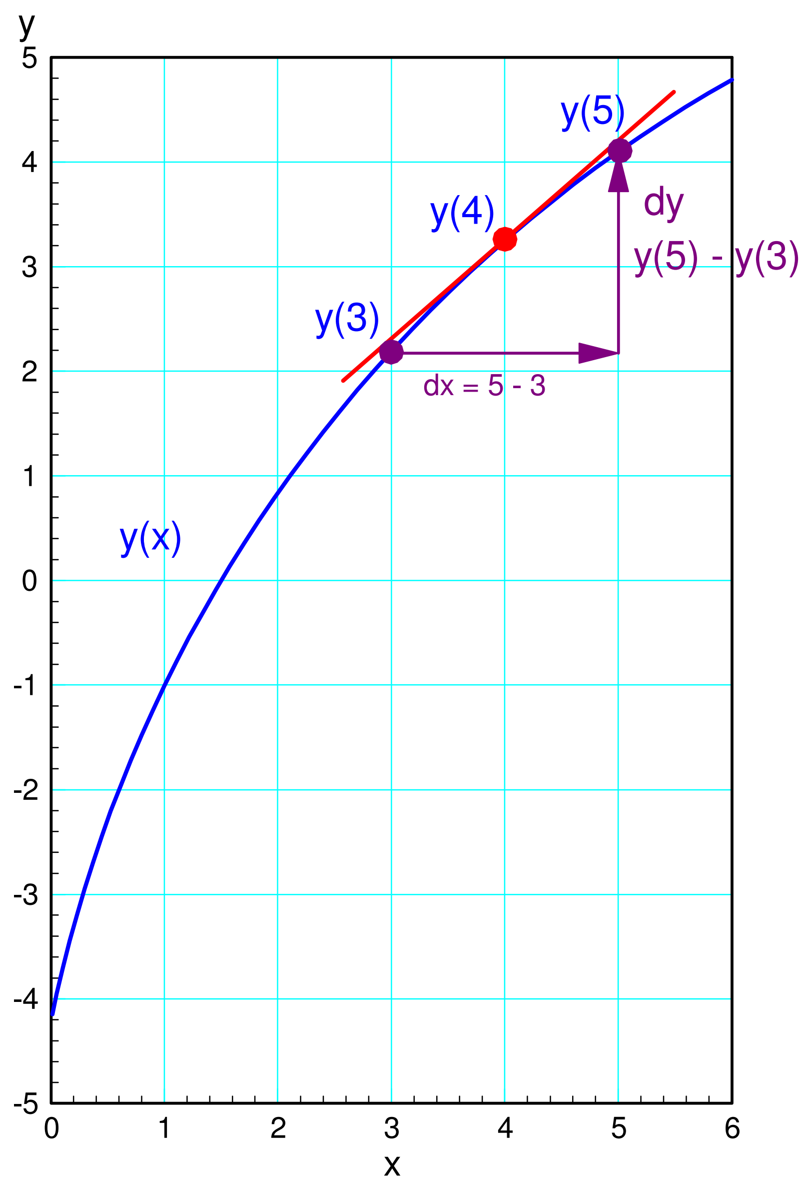
Differentiation Method #3: Numerical Differentiation.

Repeating, the derivative is

$$\frac{dy}{dx} \equiv \frac{\text{the change in y}}{\text{the change in x}}$$

You can approximate this as

$$\left(\frac{dy}{dx}\right)_{x=4} \approx \frac{y(5)-y(3)}{x(5)-x(3)}$$



Repeat a a bunch of points

$$y'(i) \approx \frac{y(i+1)-y(i-1)}{x(i+1)-x(i-1)}$$

Note: Endpoints cause problems

- Can't go beyond the endpoints

Use a slightly different equation for the endpoints

$$y(1) \approx \frac{y(2)-y(1)}{x(2)-x(1)}$$

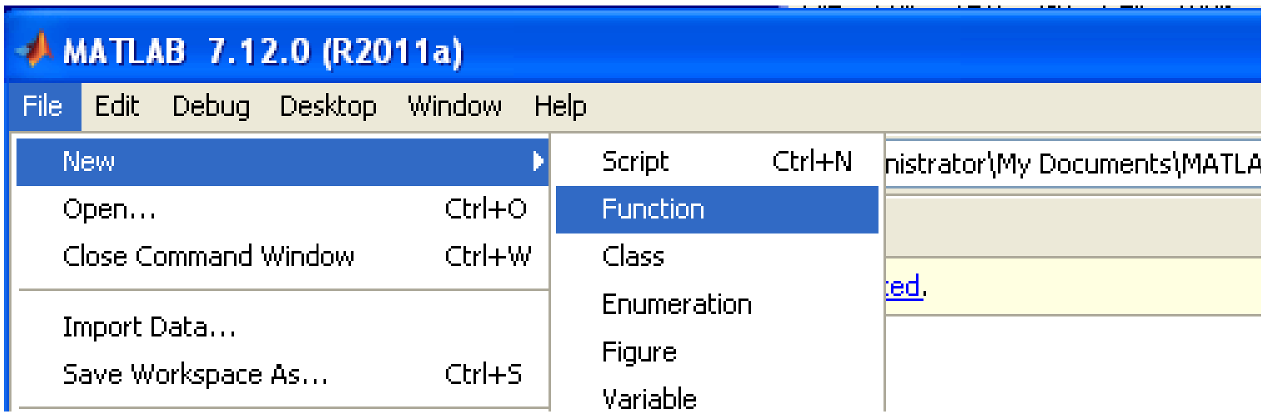
$$y'(n) \approx \frac{y(n)-y(n-1)}{x(n)-x(n-1)}$$

Differentiation in Matlab

Create a function

- A Matlab routine you can call
- Similar to cos(x) or exp(x)

In Matlab, click on 'File New Function'

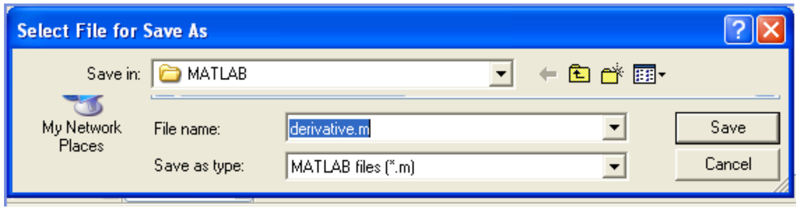


Creating A New Function in Matlab: File New Function

In the editor window, type in the following:

```
function [dy ] = derivative( x, y )

npt = length(x);

dy = 0*x;

for i=2:npt-1
   dy(i) = ( y(i+1) - y(i-1) ) / ( x(i+1) - x(i-1) );
end

dy(1) = (y(2) - y(1)) / ( x(2) - x(1) );
dy(npt) = (y(npt) - y(npt-1)) / ( x(npt) - x(npt-1));

end
```

Save as 'deravitive.m'



Check your function

- Always a good idea
- Check against a known answer

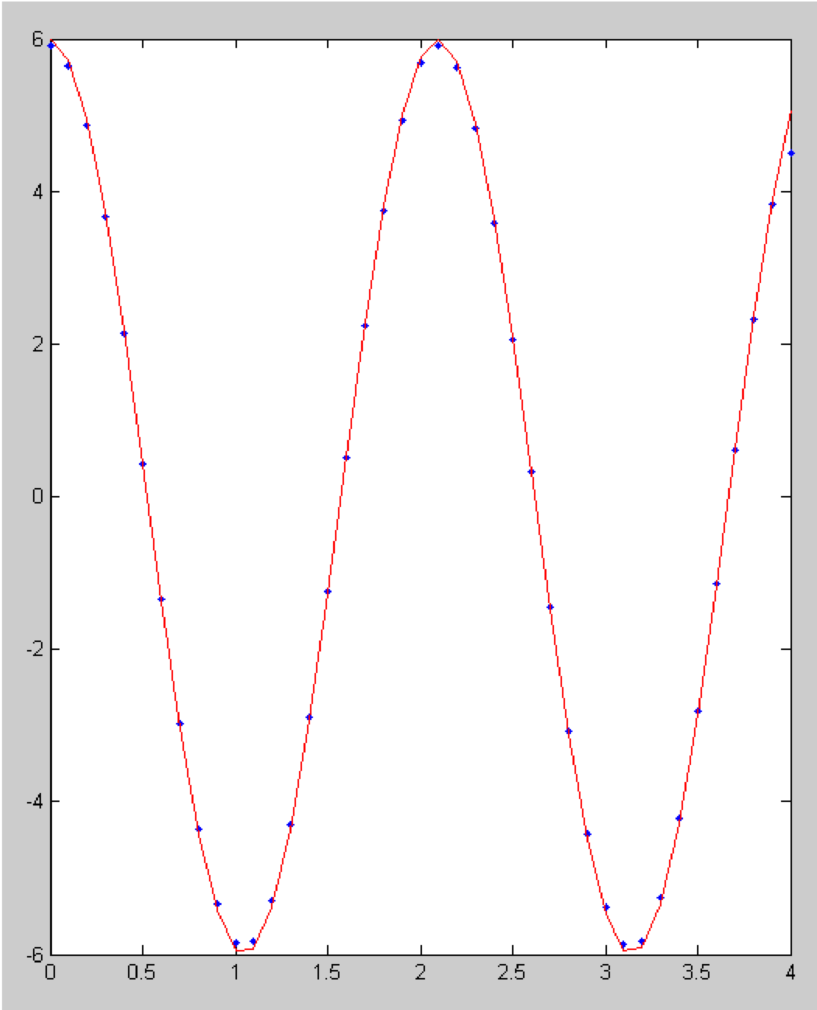
From Math 165,

$$\frac{d}{dx}(2\sin(3t)) = 6\cos(3t)$$

Checking in Matlab:

```
>> x = [0:0.1:4]';
>> y = 2*sin(3*x);
>> dy = derivative(x,y);
>> plot(x,dy,'b.',x,6*cos(3*x));
```

c



The two match up

- The derivative function seems to be working
- The endpoints are a little off (common problem)

Example 2:

- Numerical solutions work even if you don't know the answer

Find the derivative of

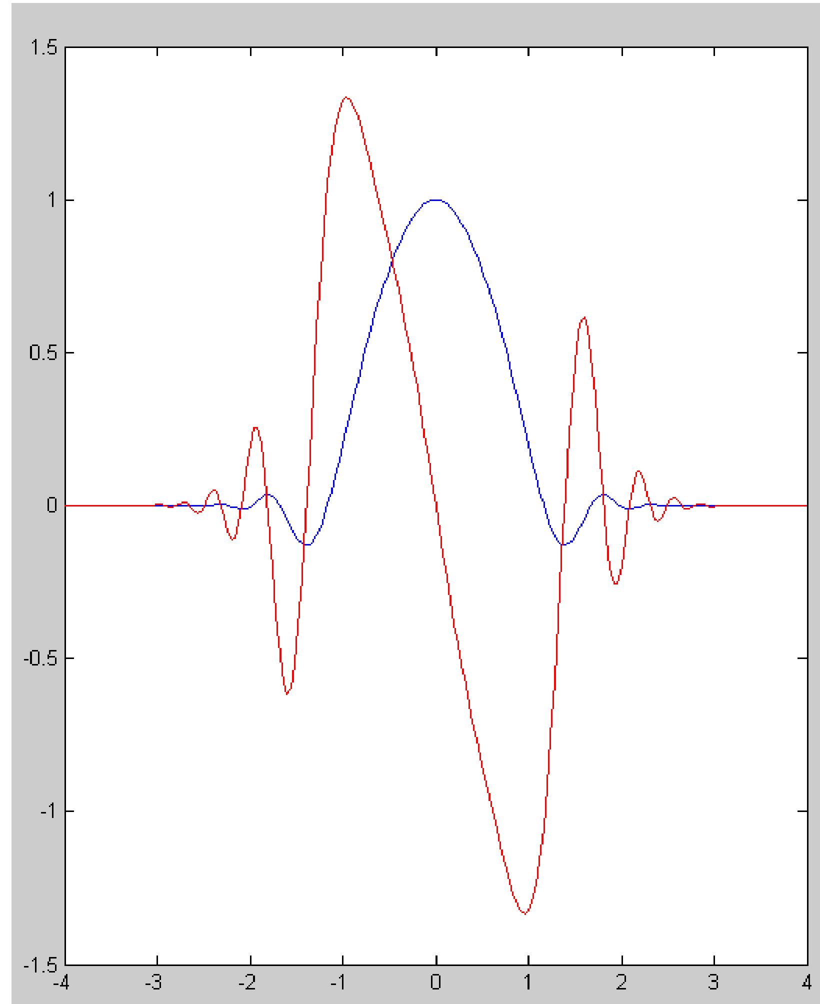
$$y = e^{-x^2} \cdot \cos(x^3)$$

First, input y(x) into Matlab

```
>> x = [-4:0.04:4]';
>> y = exp(-x.^2) .* cos( x.^3);
>> plot(x,y)
```

Now find the derivative: dy/dx

```
>> dy = derivative(x,y);
>> plot(x,y,'b',x,dy,'r')
```

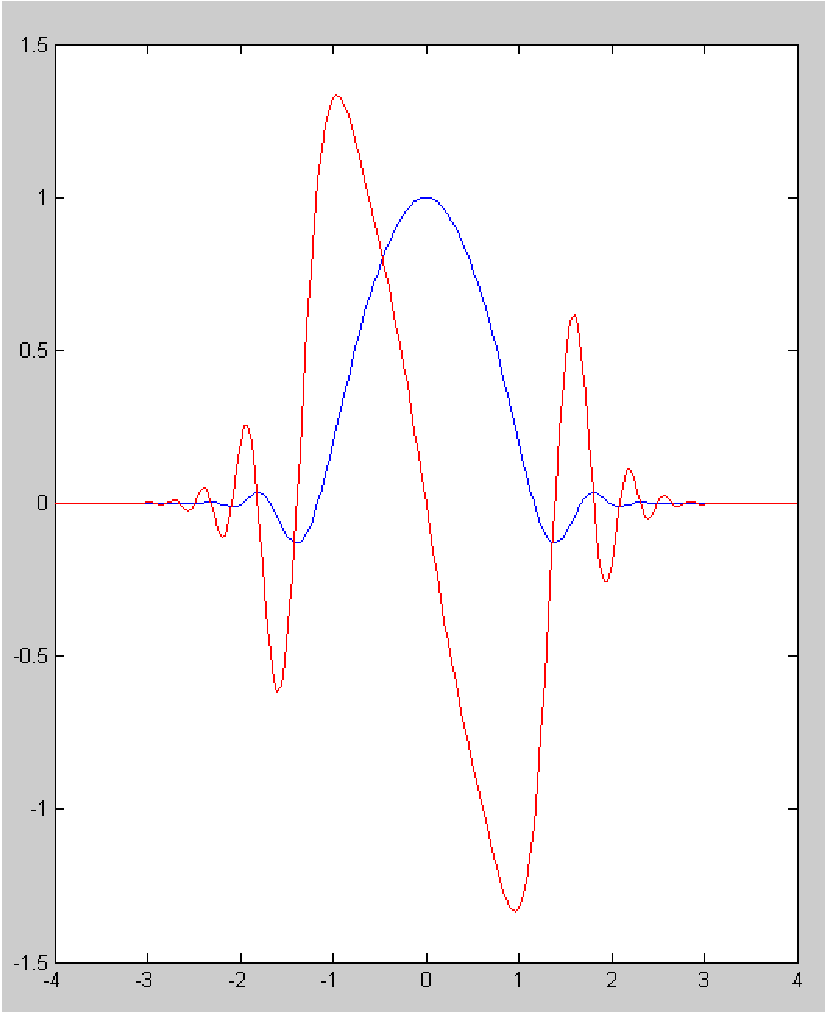


Finding local maximums & minimums

y(x) is a max/min when y'(x) = 0

This is how you'll find the max/min of a funciton in Math 165

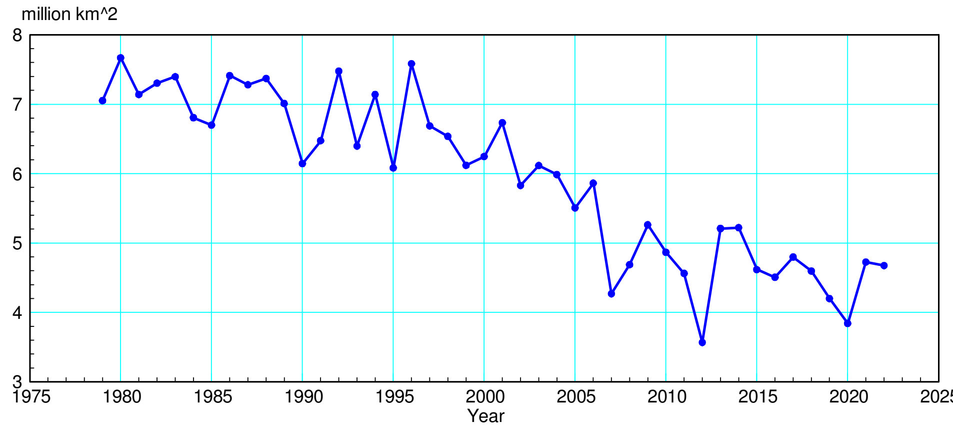
- Take the derivative
- Set y'(x) = 0



Differentiation and Noise

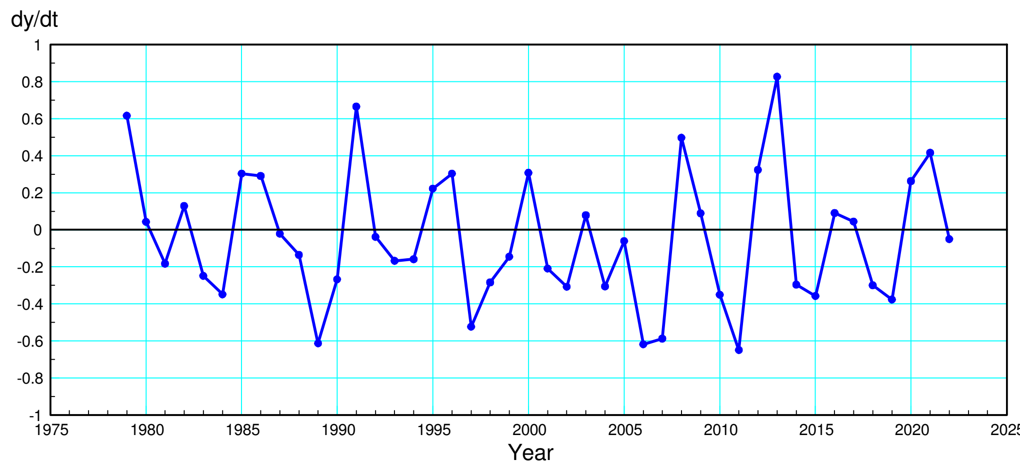
Differentiation amplifies noise

- Major problem



Arctic Sea Ice (top curve)

- Decent data
- Seems like a downward trend



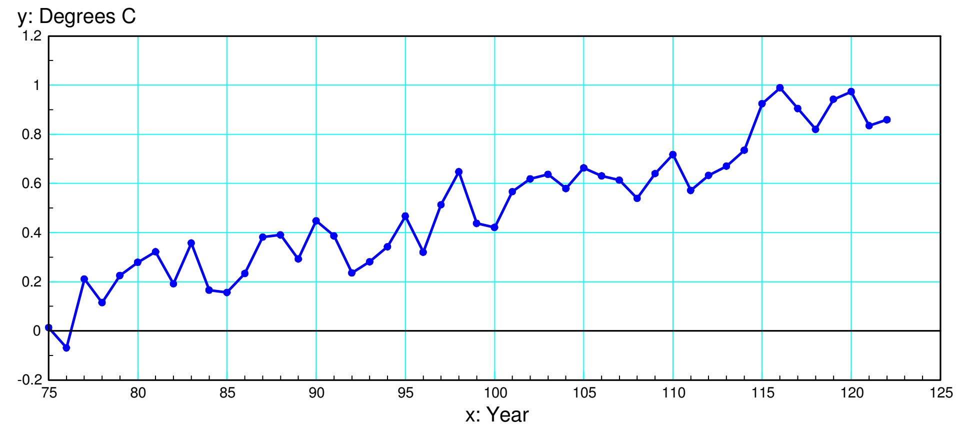
Derivative (bottom curve)

- Lots of noise
- Can't tell what's happening

Example 2:

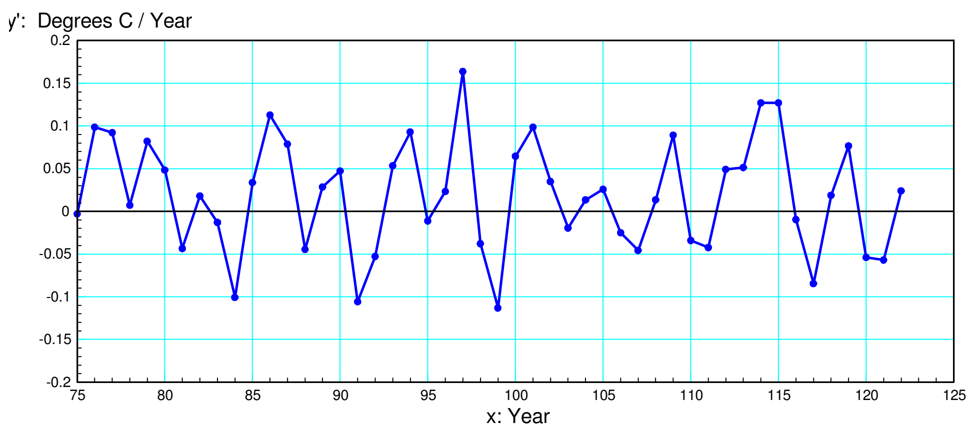
Global Temperatures (top curve)

- Decent data
- Seems to be an upward trend



Derivative (bottom curve)

- Looks like random noise
- Can't see the trend

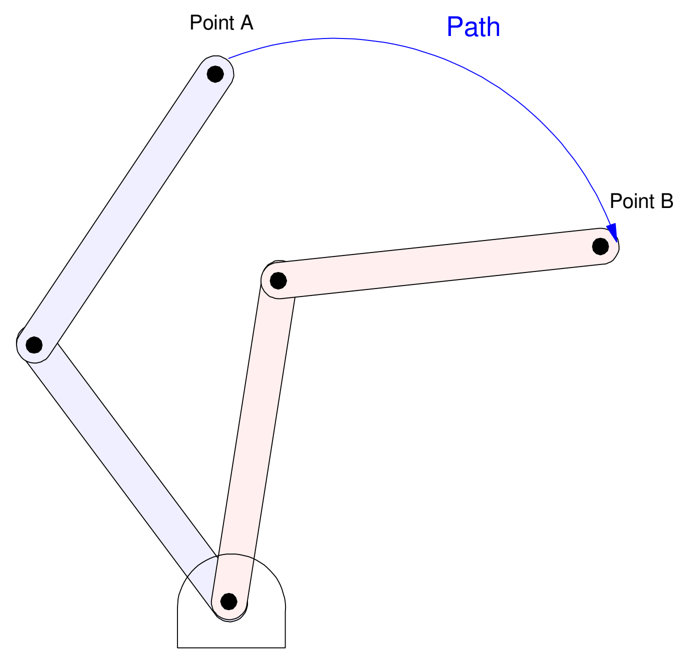


Robotics & Path Planning

Another application of derivatives.

Problem:

- Find a path from point A to B
- Keep the 1st and 2nd derivatives small



Why?

- y(x) = Motor angle
- 1st derivative: Motor velocity
  - Roughly same as voltage for a DC motor
- 2nd derivative: Motor acceleration
  - Roughly same as current for a DC motor

Option 1: Linear Motion.

Assume motor follows the path

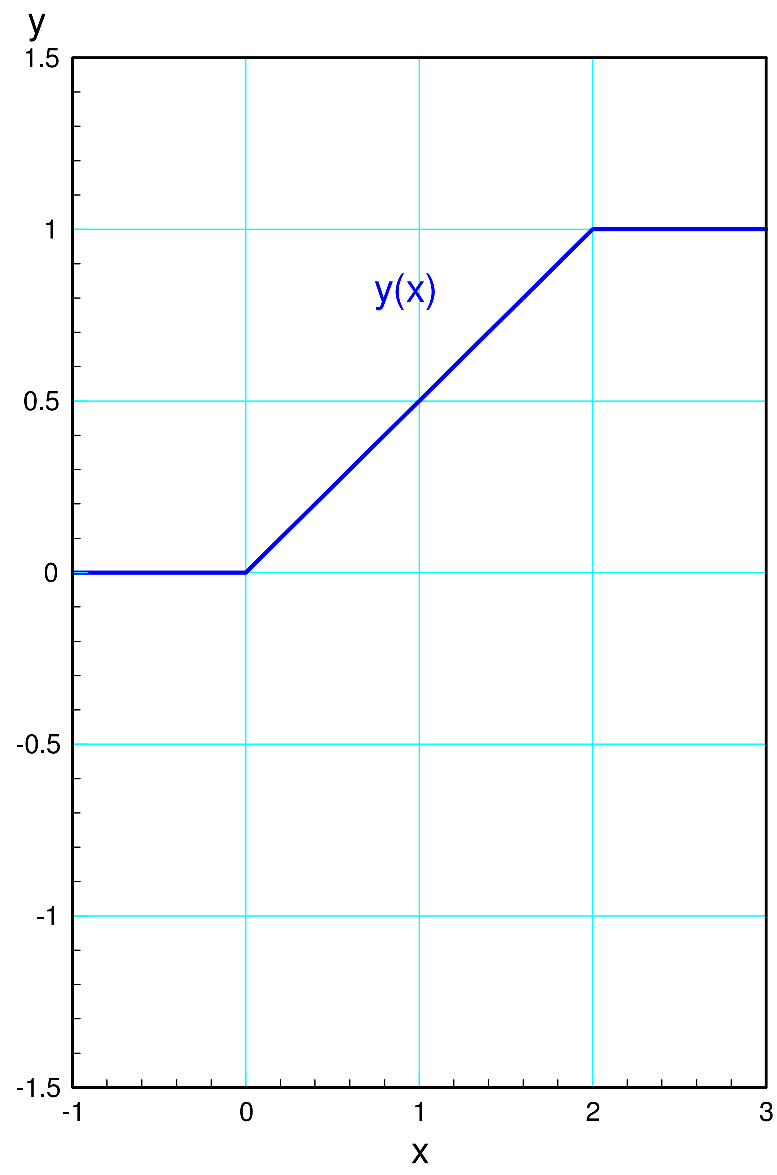
$$y = \begin{Bmatrix} 0 & t < 0 \\ t/2 & 0 < t < 2 \\ 1 & t > 2 \end{Bmatrix}$$

Advantage:

- Simple function

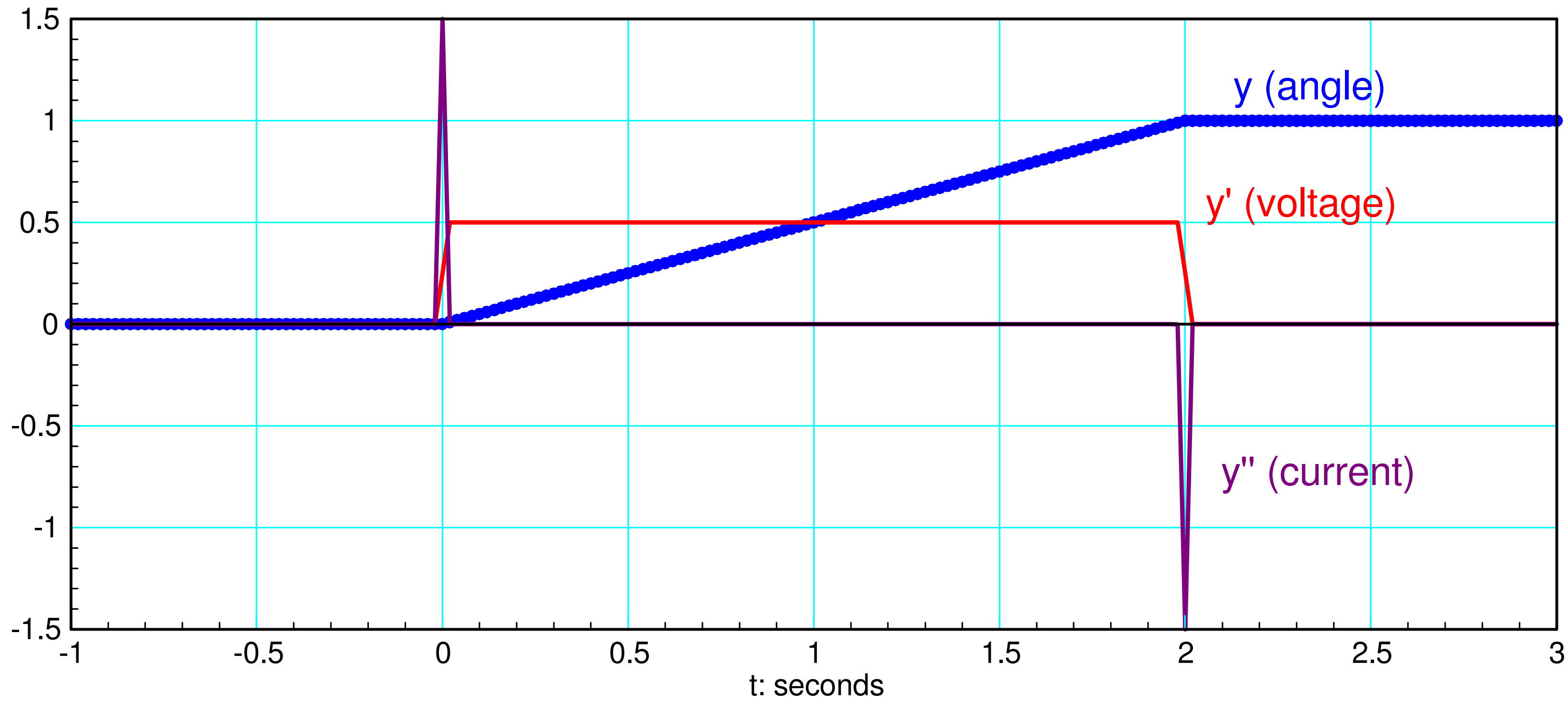
Disadvantage:

- Jump discontinuity in voltage (y')
- Current goes to infinity (y'')



Find the derivatives in Matlab

```
>> t = [-1:0.01:3]' + 1e-6;
>> y = 0*(t<0) + (t/2) .* (t>0).*(t<2) + (1)*(t>2);
>> dy = derivative(t,y);
>> ddy = derivative(t,dy);
>> plot(t,y,t,dy,d,ddy)
```



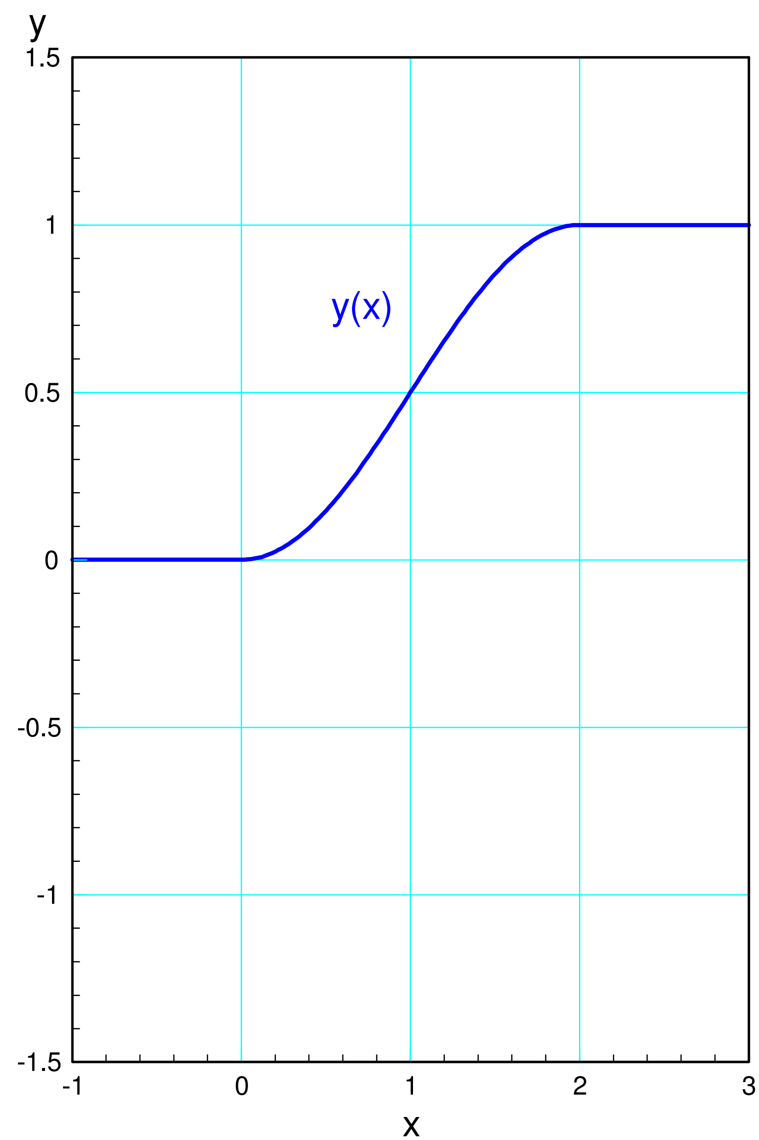
Option 2: Cosine Motion

Let

$$y = \begin{cases} 0 & t < 0 \\ \left(\frac{1}{2}\right)\left(1 - \cos\left(\frac{\pi t}{2}\right)\right) & 0 < t < 2 \\ 1 & t > 2 \end{cases}$$

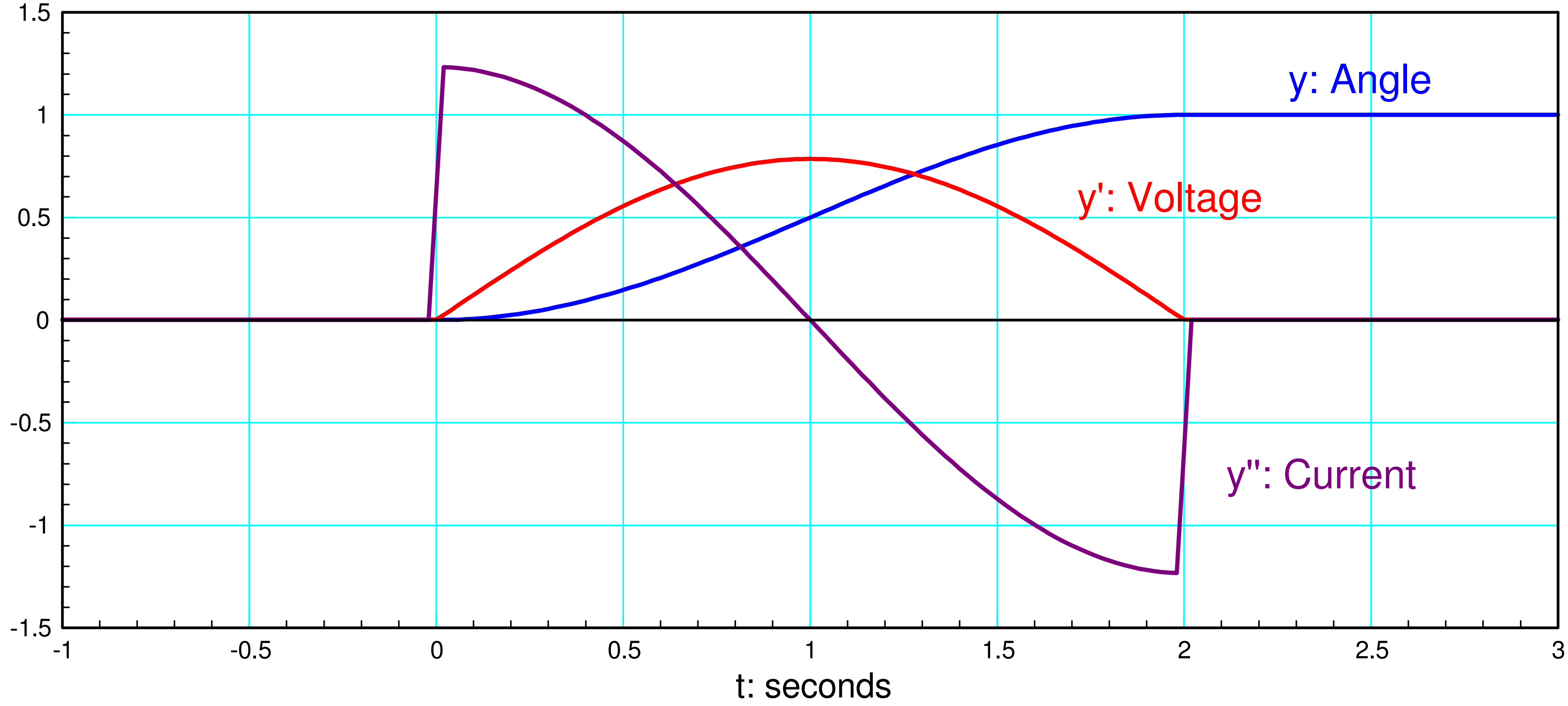
A little more complicated function

- 1st derivative is finite
- 2nd derivative is finite



In Matlab:

```
>> t = [-1:0.01:3]' + 1e-6;
>> y = 0*(t<0) + ((1-cos(pi*t/2))/2) .* (t>0).*(t<2) + (1)*(t>2);
>> dy = derivative(t,y);
>> ddy = derivative(t,dy);
>> plot(t,y,t,dy,t,ddy)
```



Other paths from A to B can be defined

- Keep acceleration constant
- Start and end with zero acceleration
- Cubic function for y(x)
- Other

This is kind of the idea with path planning

- What's the best path for going from point A to point B?
- Derivatives are used to evaluate each path

Summary:

Derivatives and differentiation are useful:

- Bank Account
  - Derivative tells you how much money is being deposited or withdrawn
- Arctic Sea Ice
  - How much ice is being added (positive derivative
  - How much ice is being lost (negative derivative)
- DC Motors
  - Voltage to the motor (1st derivative)
  - Current to the motor (2nd derivative)

Graphical methods work for any function

- The derivative is the slope at any point

Numeric methods work

- If you can get the function into Matlab, you can find the derivative
- Noise causes problems: differentiation amplifies noise