Signals and Systems
Filters

ECE 111 Introduction to ECE

Jake Glower - Week #14

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Filters:

A filter is any circuit whose gain varies with frequency

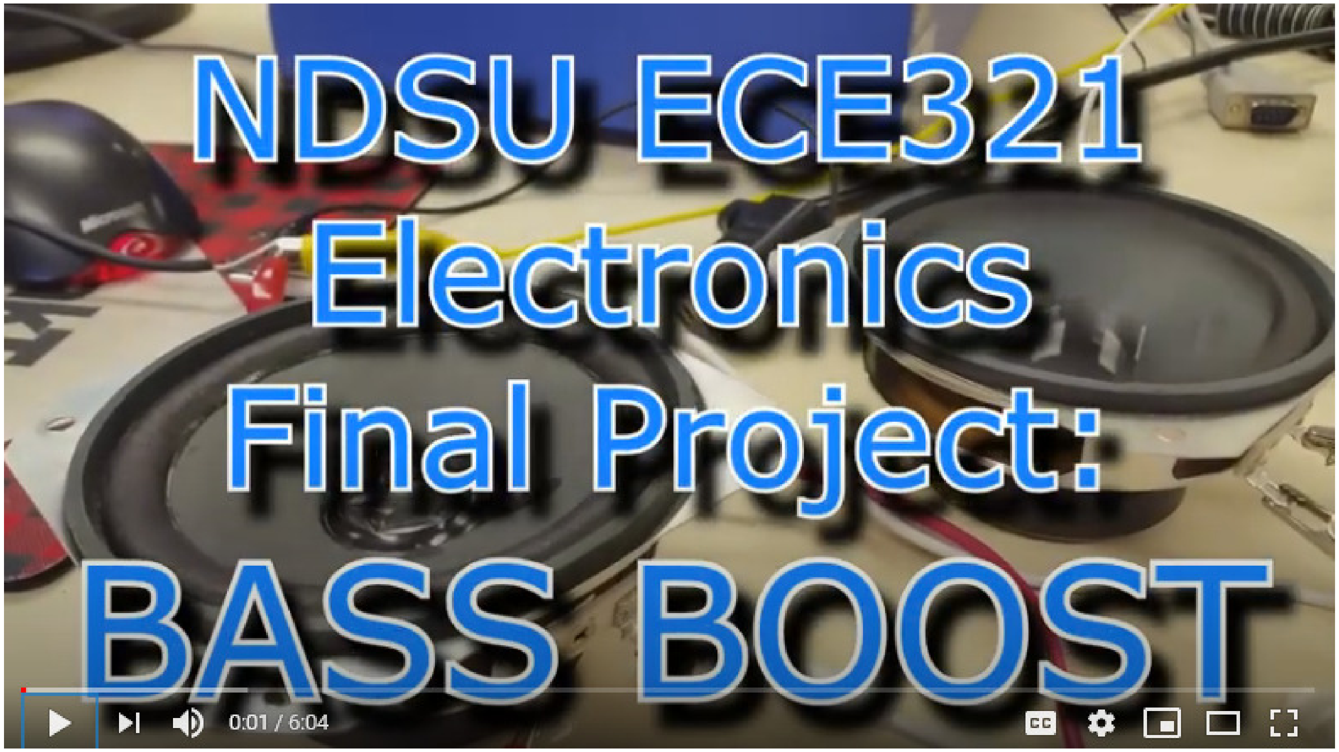
- Any circuit with inductors and/or capacitors
- Anything that satisfies a differential equation

Filter design looks at how to choose the filter to

- Pass frequencies you want, and
- Reject frequencies you don't want.

Example: Bass Boost

- https://www.youtube.com/watch?v=zKfc_VoyVUM&feature=youtu.be
- Building a sub-woofer crossover
- Pass frequencies below 250Hz
- Reject frequencies above 400Hz



Differential Equations

Differential equations describe almost everything

- Why Calculus I, II, III, IV are required

Any circuit with inductors and capacitors are described by differential equations

Inductor:

$$E = \tfrac{1}{2} L I^2$$

$$\tfrac{d}{dt}(E) = P = VI = LI \tfrac{dI}{dt}$$

$$\boxed{V = L \frac{dI}{dt}}$$

Capacitor:

$$E = \tfrac{1}{2} C V^2 \qquad \text{Joules}$$

$$\tfrac{d}{dt}(E) = VI = CV \tfrac{dV}{dt} \qquad \text{Watts}$$

$$\boxed{I = C \frac{dV}{dt}}$$

Transfer Functions

Assume a 3rd-order differential equation relating x and y:

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

$$y' \equiv \frac{dy}{dx}$$

Assume all functions are in the form of

$$y(t) = e^{st}$$

Then

$$\frac{d}{dt}(e^{st}) = s \cdot e^{st}$$

sY means *the derivative of y(t)*

With this assumption

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

becomes

$$s^3 Y + 4s^2 Y + 6sY + 8Y = 10sX + 30X$$

Solving for Y:

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right) X = G(s)\, X$$

G(s) is called the *transfer function* of the system.

- Essentially, it is the gain from X to Y

Example: Find the differential equation relating X and Y given

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)X$$

Solution: First, cross multiply

$$(s^3 + 4s^2 + 6s + 8)Y = (10s + 30)X$$

Next, replace each 's' with $\frac{d}{dt}$

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

or equivalently

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 10\frac{dx}{dt} + 30x$$

Handout

Problem 1: Determine the transfer function from the differential equation

$$y'' + 5y' + 8y = 2x' + 10x$$

Handout

Problem 2: Determine the differential equation which relates X and Y

$$Y = \left(\frac{10s+20}{s^2+6s+5}\right)X$$

Transfer Functions with DC:

Find y(t):

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)X \qquad\qquad x(t) = 2$$

Solution:

$$x(t) = 2 \cdot e^{0t} = 2$$

$$s = 0$$

$$X = 2 + j0$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)_{s=0}(2 + j0) = 7.50$$

$$y(t) = 7.5$$

Transfer Function with a Sinusoidal Input

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right) X \qquad x(t) = 2\cos(3t)$$

Convert to phasor form

$$s = j3$$

$$X = 2 + j0 \qquad a + jb \to a \cdot \cos(\omega t) - b \cdot \sin(\omega t)$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)_{s=j3} \cdot (2 + j0)$$

$$Y = -2.566 - j1.318 \qquad \textit{rectangular form}$$

$$Y = 2.885 \angle -152.8^0 \qquad \textit{polar form}$$

meaning

$$y(t) = -2.566\cos(3t) + 1.318\sin(3t)$$

$$y(t) = 2.885\cos(3t - 152.8^0)$$

Either form is valid

Note: Answer varies with frequency

- It's a filter

Example: Find y(t) for an input at 30 rad/sec:

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)X \qquad x(t) = 2\cos(30t)$$

Solution:

$$s = j30$$

$$X = 2 + j0$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)_{s=j30} \cdot (2 + j0)$$

$$Y = (-0.0223 - j0.0007)$$

which means

$$y(t) = -0.0223\cos(30t) + 0.0007\sin(30t)$$

MATLAB Code:

Input the frequency for s and evaluate G(s)

```
s = 0;
X = 2;
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2)

Y =    7.5000

s = j*3;
X = 2 + j*0;
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2 + j*0)

Y =  -2.5665 - 1.3179i

s = j*30;
X = 2 + j*0;
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2 + j*0)

Y =  -0.0223 - 0.0007i
```

You can also input G(s) as a transfer function and use the MATLAB function evalfr()

```
G = tf([10,30],[1,4,6,8])

      10 s + 30
---------------------
s^3 + 4 s^2 + 6 s + 8

Y = evalfr(G, 0) * 2

Y =    7.5000

Y = evalfr(G, j*3) * 2

Y =  -2.5665 - 1.3179i

Y = evalfr(G, j*30) * 2

Y =  -0.0223 - 0.0007i
```

which are the same answers as before.

Handout

Problem 3: Find y(t)

$$Y = \left(\frac{10}{(s+1)(s+3)}\right) X$$

$$x(t) = 4\cos(5t) + 2\sin(5t)$$

Handout

$$Y = \left(\frac{10}{(s+1)(s+3)}\right)X \qquad x(t) = 4\cos(5t) + 2\sin(5t)$$

Answer:

$$s = j5$$

$$X = 4 - j2$$

$$Y = \left(\frac{10}{(s+1)(s+3)}\right)_{s=j5}(4 - j2)$$

$$Y = -1.448 - j0.407$$

meaning

$$y(t) = -1.448\cos(5t) + 0.407\sin(5t)$$

Frequency Response of a Filter:

- If the input is known, plug in s = jw
- For a general solution, sweep w

Example: Determine the gain of G(s) over the range of 0 to 10 rad/sec for

$$G(s) = \left(\frac{10s+30}{s^3+4s^2+6s+8}\right)$$

Option 1: Compute the gain at a bunch of points from 0 to 10 rad/sec, or

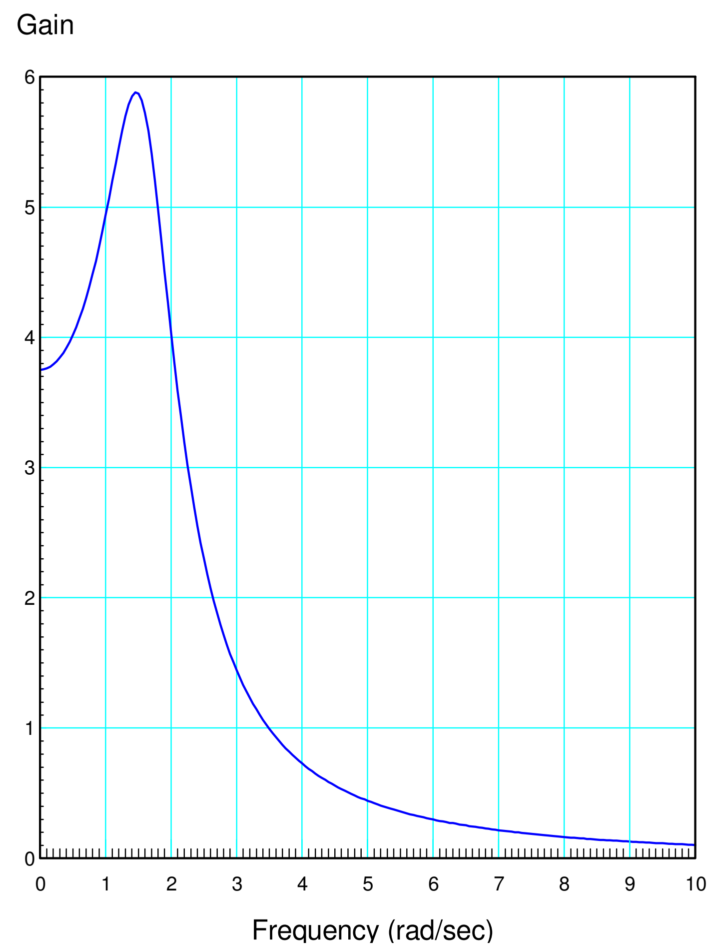
Option 2: Use MATLAB. Input the frequencies you want to evaluate:

```
w = [0:5:20]';
s = j*w;
G = (10*s + 30) ./ (s.^3 + 4*s.^2 + 6*s + 8);
```

Note that dot-notation is required.

In matlab:

```
w = [0:0.05:10]';
s = j*w;
G = (10*s + 30) ./ (s.^3 + 4*s.^2 + 6*s +
8);
plot(w,abs(G));
xlabel('Frequency (rad/sec)');
ylabel('Gain');
```



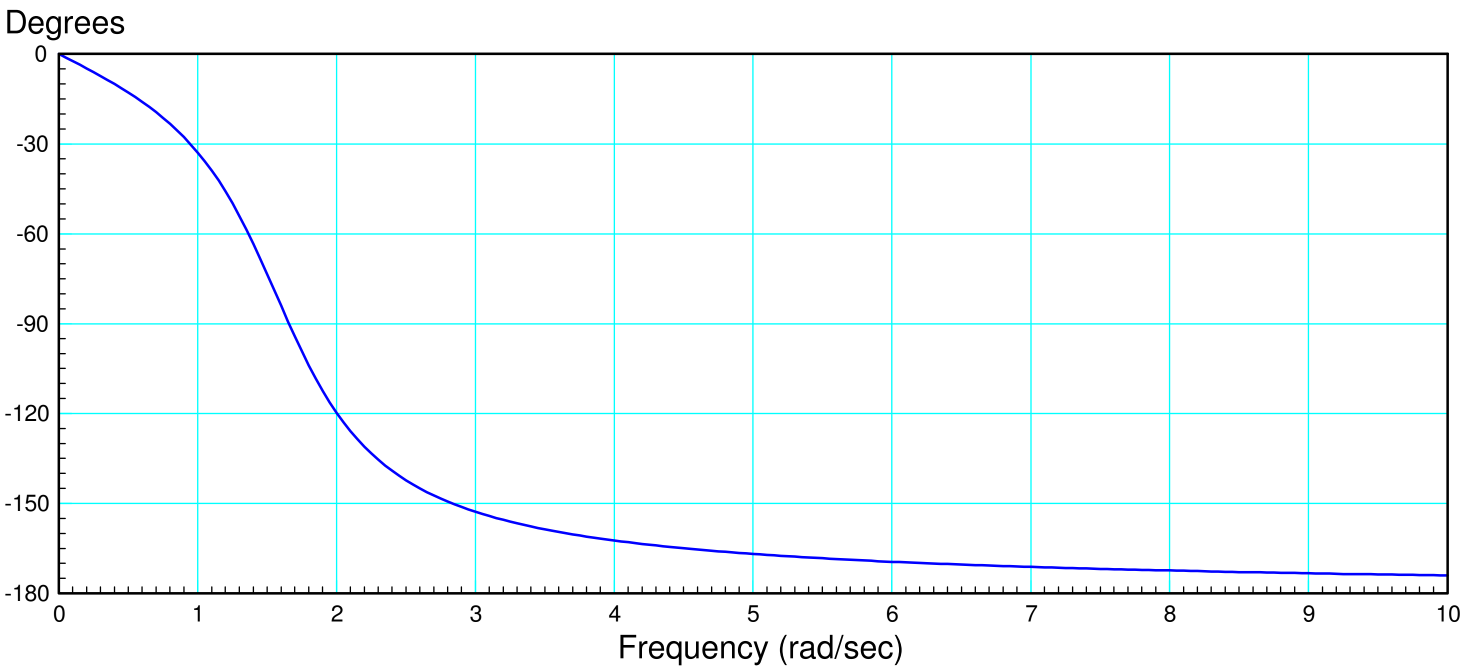
What this graph tells you is:

- The gain is large for frequencies below about 2 rad/sec and small for frequencies above 6 rad/sec. Since this passes low frequencies, it is called *a low-pass filter*
- The system has a resonance (a large gain) for frequencies near 1.5 rad/sec.

Phase Plot

- Not sure what this really tells you
- Usually we only deal with amplitude (gain)

```
>> plot(w,angle(G)*180/pi);
>> xlabel('Frequency (rad/sec)');
>> ylabel('Angle (degrees)');
```

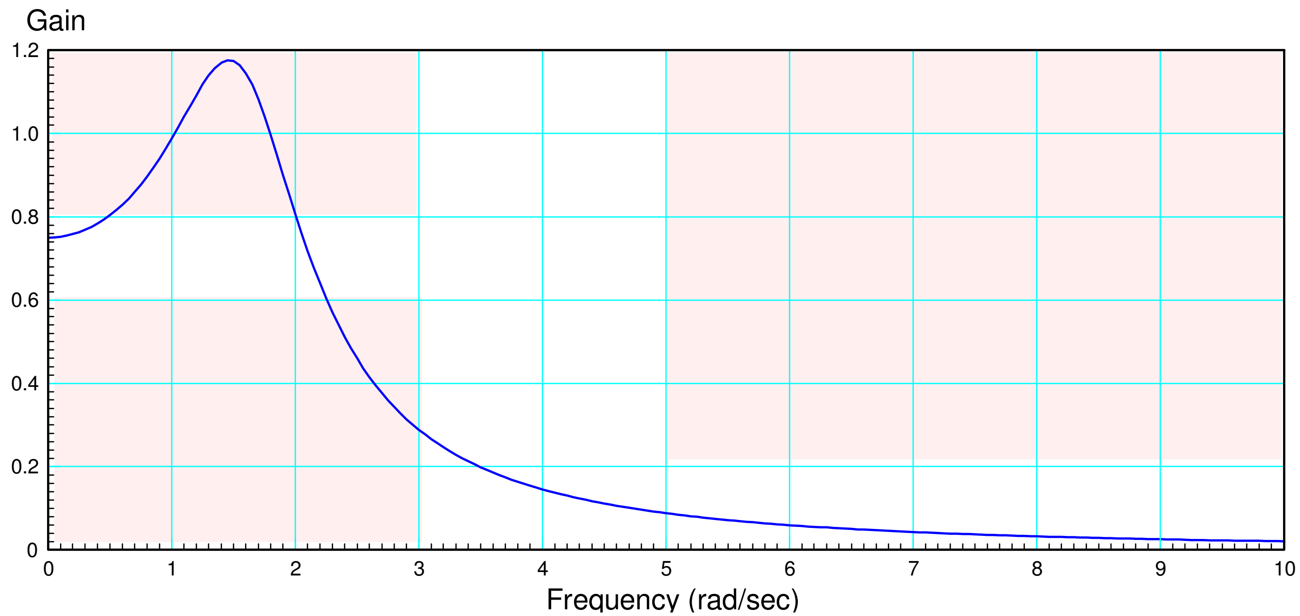


fminsearch() and m-files

Problem: How to design a filter?

- What is a 'good' transfer function?
- Covered in ECE 343 & ECE 321

Knowing nothing about filter design, you can still design a filter using Matlab



fminsearch()

- Really useful Matlab function
- Finds the minimum of a function

Example: Find $\sqrt{2}$

```
function [ J ] = cost( z )
   e = z*z - 2;
   J = e^2;
end
```

Minimize in Matlab

```
>> [a,b] = fminsearch('cost',4)

a =     1.4143
b =     1.5665e-008
```

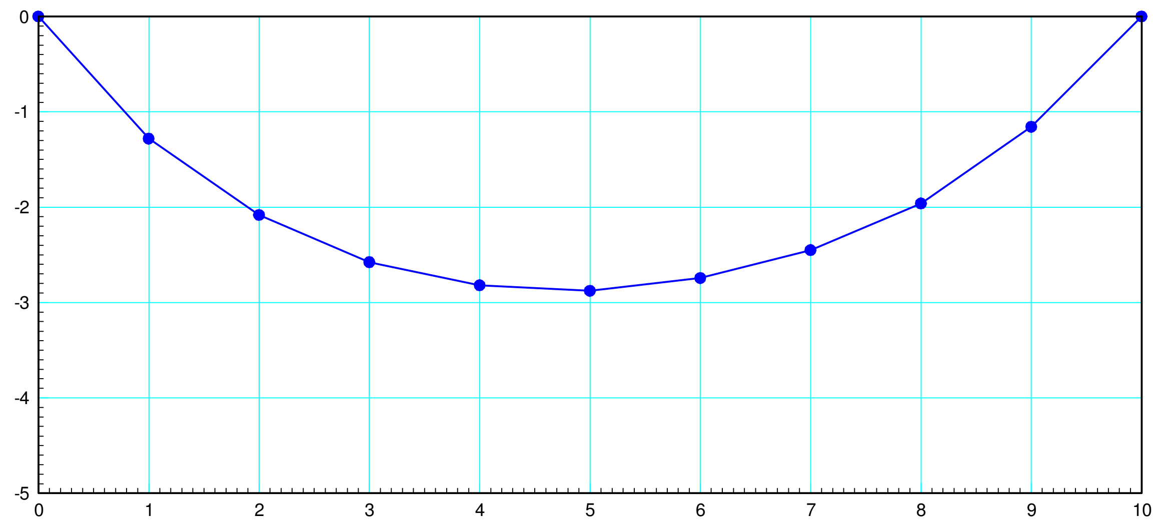
Example: Shape of a hanging chain

Minimize the potential energy

$$PE = mg(y_1 + y_2 + ... + y_9)$$

Constrain the length to be 12 meters (ish)

$$J = PE + \alpha(12 - L)^2$$




```
function [ J ] = cost_chain( Z )

   Y = [0;Z;0];
   PE = sum(Y);
   L = 0;
   for i=2:11
      L = L + sqrt(1 + (Y(i) - Y(i-1))^2);
   end

   E = (12 - L);
   J = PE + 100*E^2;

   plot([0:10],Y,'.-');
   ylim([-5,1]);
   pause(0.01);

   end


y = i .* (i-10) / 5;
[a,b] = fminsearch('cost',y)
```

Filter Design with fminsearch:

$$|G_d(s)| = \begin{cases} 1 & \omega < 3 \\ 0 & \omega > 3 \end{cases}$$

Step 1: Assume the form of the filter

$$G(s) = \left(\frac{a}{\left(s^2+bs+c\right)\left(s^2+ds+e\right)} \right)$$

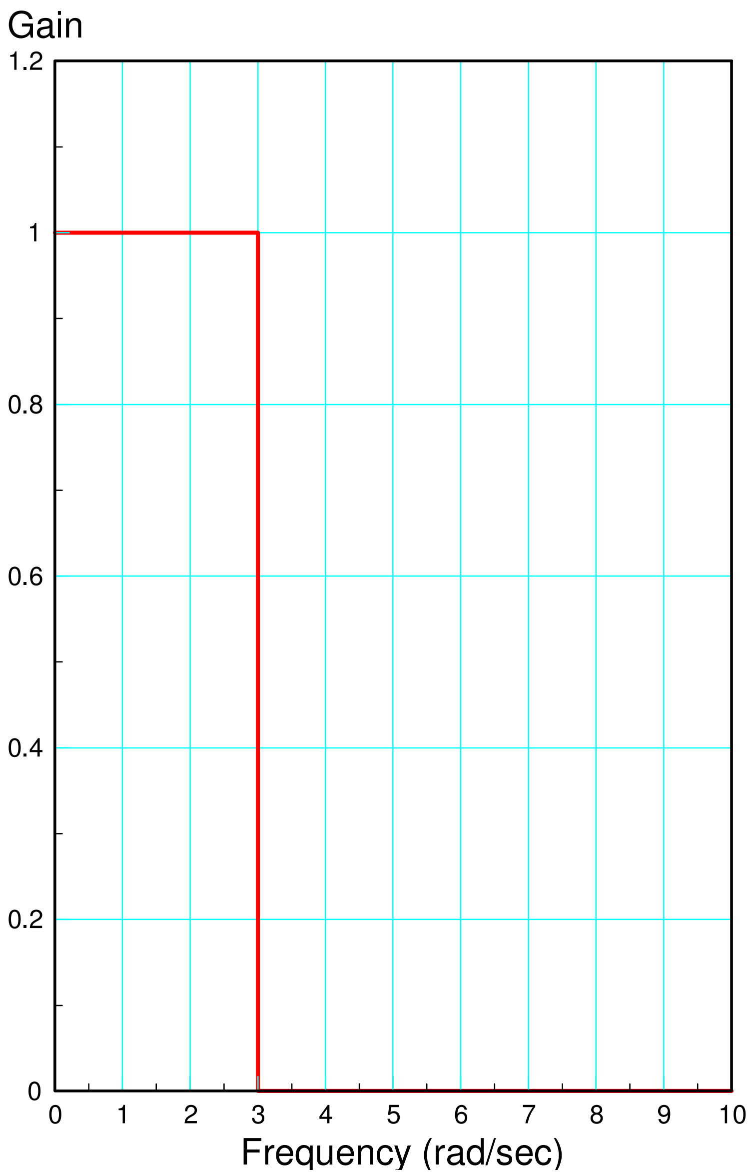
Define the cost (J)

- Minimum is when G(s) = desired filter

$$E(s) = |G(s)| - |G_d(s)|$$

$$J = \sum E^2$$

Guess {a, b, c, d, e} to minimize J

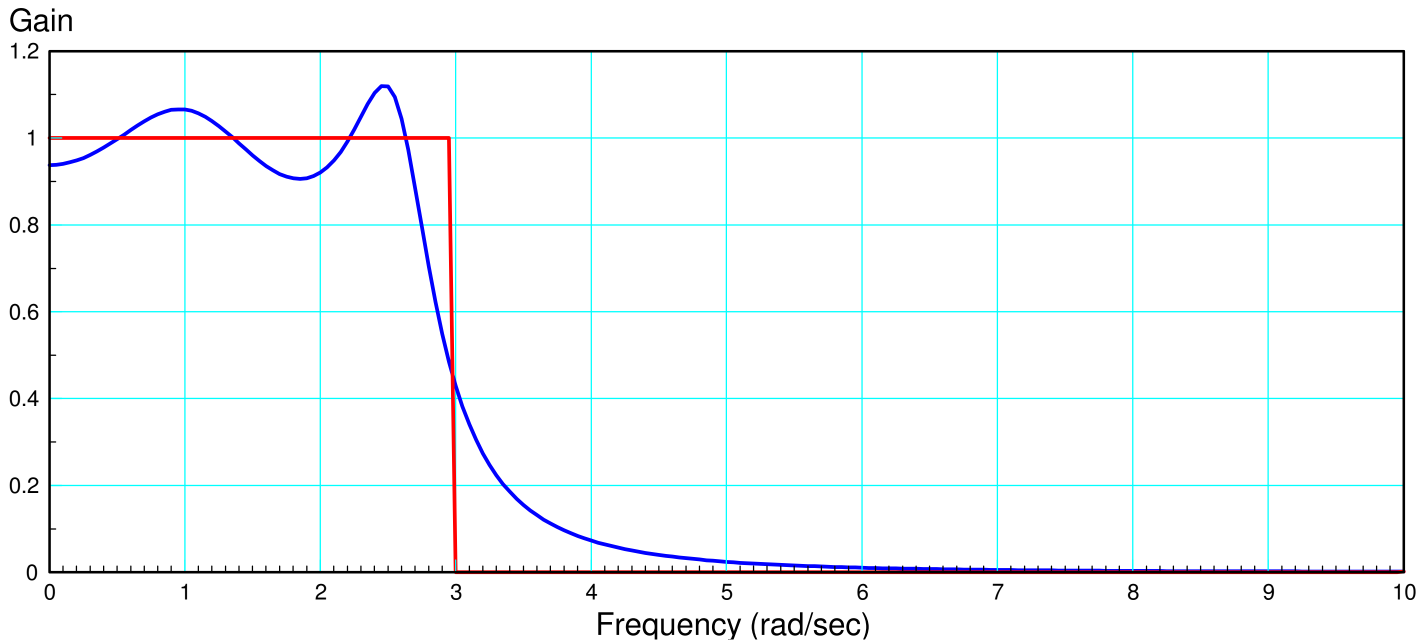


```
function [ J ] = costF( z )

   a = z(1);
   b = z(2);
   c = z(3);
   d = z(4);
   e = z(5);

   w = [0:0.1:10]';
   s = j*w;

   Gideal = 1 * (w < 3);

   G = a ./ ( (s.^2 + b*s + c).*(s.^2 + d*s + e) );

   e = abs(Gideal) - abs(G);

   J = sum(e .^ 2);

   plot(w,abs(Gideal),w,abs(G));
   ylim([0,1.2]);
   pause(0.01);

end
```

Call fminsearch with an initial guess for (a,b,c,d)

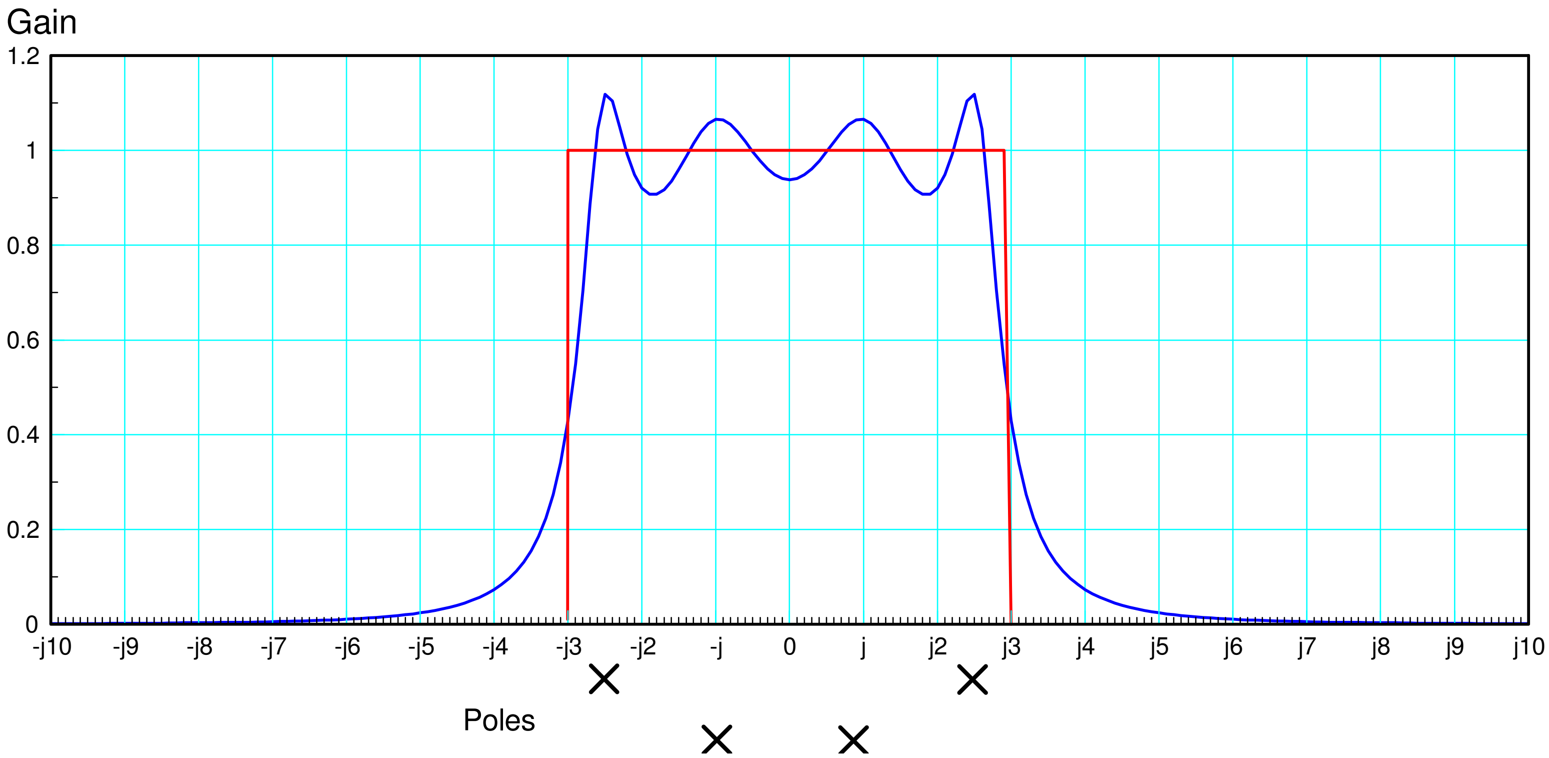
```
>> [Z,e] = fminsearch('costF',[1,2,3,4,5])
            a          b          c          d          e
Z =    10.9474    1.6224    1.7317    0.6141    6.7413
e =     0.9575
```

$$G(s) = \left(\frac{10.9474}{\left(s^2+1.6224s+1.7317\right)\left(s^2+0.6141s+6.7413\right)} \right)$$



Sidelight: The filter isn't arbitrary.

- When you're close to a zero, the gain is small (multiply by a small number)
- When you're close to a pole, the gain is large (divide by a small number)
- Poles are { $s = -0.8112 \pm j1.0362$, $s = -0.3071 \pm j2.5782$ }



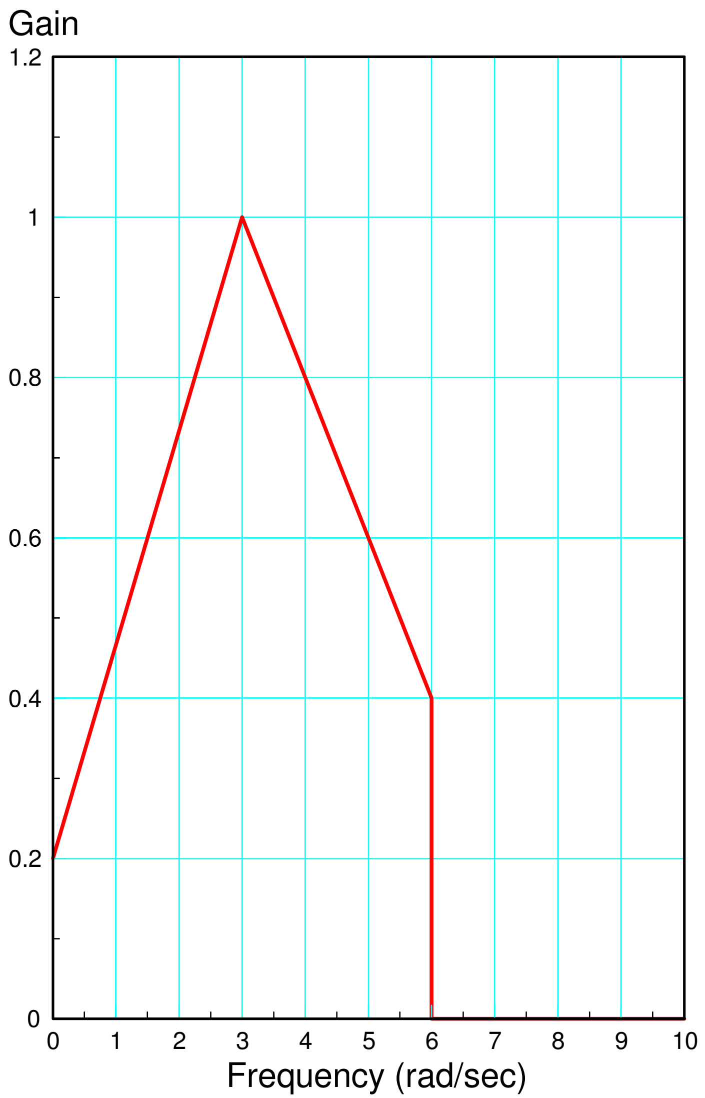
Example 2: Design a filter to match

Assume

$$G(s) = \left(\frac{a}{(s+b)\left(s^2+cs+d\right)\left(s^2+ef+g\right)} \right)$$

Use a piecewise linear model for Gideal

```
w = [0:0.1:10]';
s = j*w;

Gideal = (0.2667*w+0.2) .* (w < 3)
       + (1.6 - 0.2*w) .* (w >= 3).*(w<6);
```

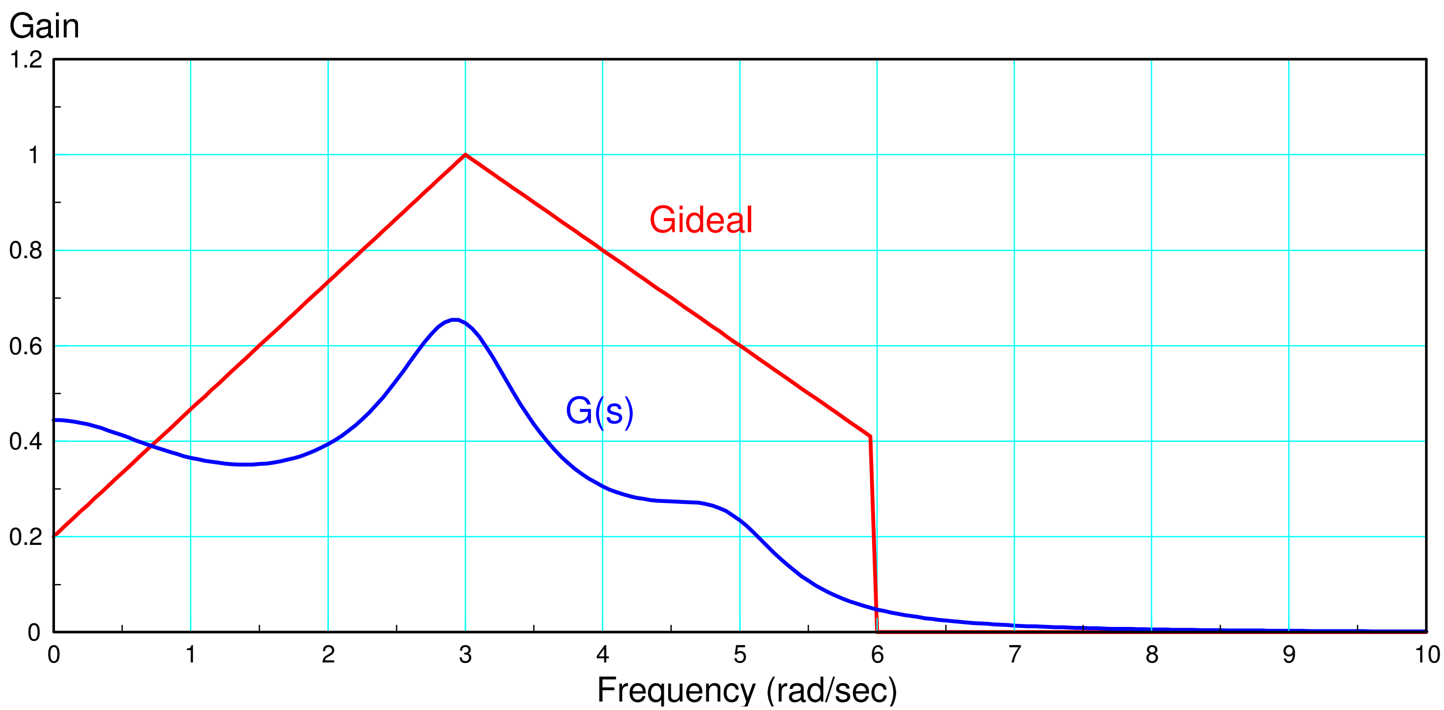


Matlab Function:

```
function [ J ] = costF( z )

   a = z(1);
   b = z(2);
   c = z(3);
   d = z(4);
   e = z(5);
   f = z(6);

   w = [0:0.1:10]';
   s = j*w;

   Gideal = (0.2667*w+0.2) .* (w<3) + (1.6 - 0.2*w) .* (w >= 3).*(w<6);

   G = a ./ ( (s+b) .* (s.^2 + c*s + d).*(s.^2 + e*s + f) );

   e = abs(Gideal) - abs(G);

   J = sum(e .^ 2);

   plot(w,abs(Gideal),w,abs(G));
   ylim([0,1.2]);
   pause(0.01);

end
```

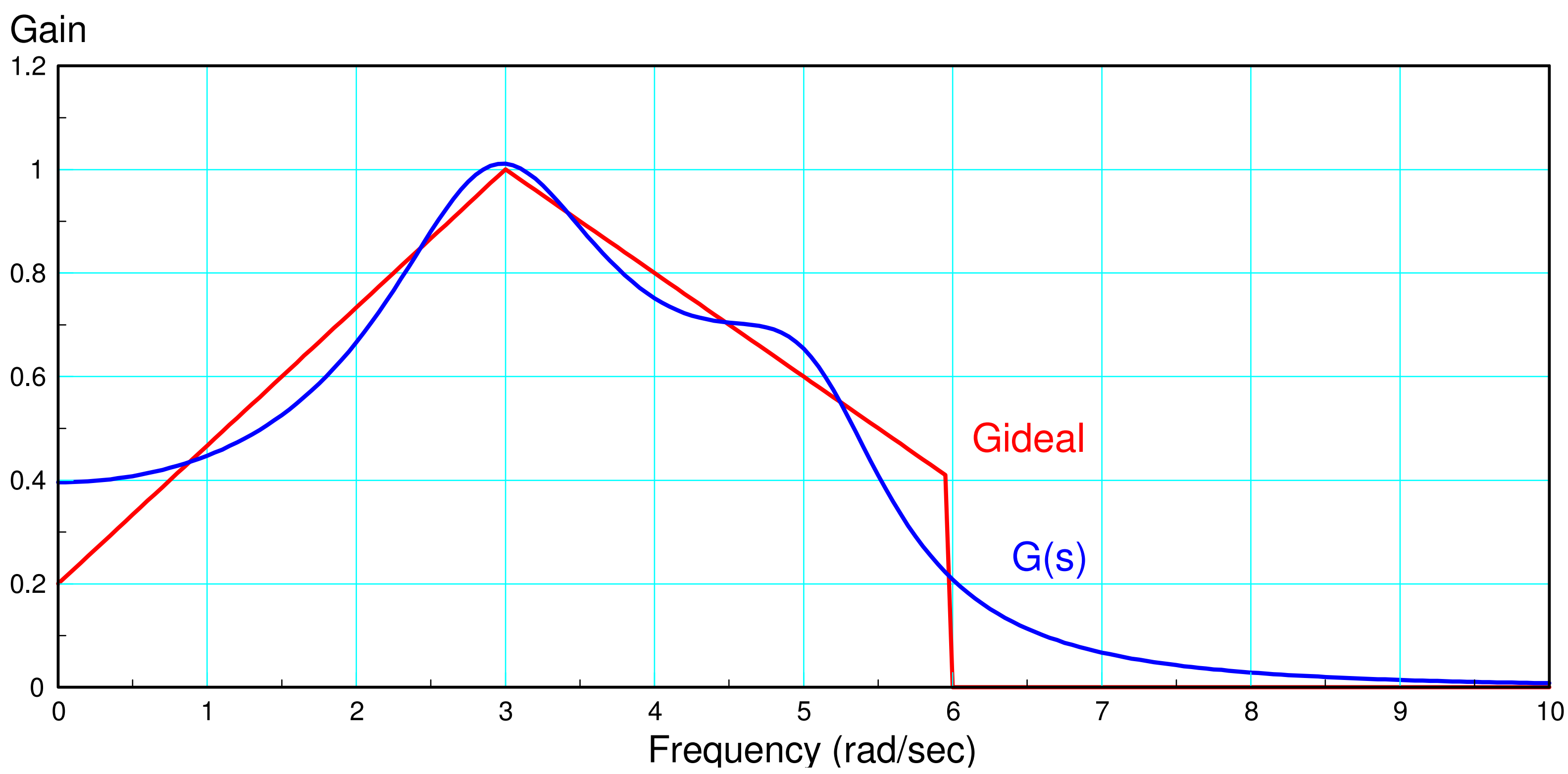
Optimization by hand

```
>> costF([1,2,3,4,5,6])
ans =   27.6268

>> costF([100,1,2,9,2,25])
ans =   13.1412

>> costF([100,1,1,9,1,25])
ans =    7.1906
```



Optimization with *fminsearch()*

```
>> [Z,e] = fminsearch('costF',[100,1,1,9,1,25])

           a            b           c          d           e         f
Z =   651.9876    6.7179    1.6175    9.2075    1.3025   26.6229

e =     0.5270
```



Summary:

A filter is a circuit where the gain depends upon frequency

- Any circuit with inductors and/or capacitors

Filter analysis is easy with complex numbers

- Plug in $s \rightarrow j\omega$
- Use phasors to represent the input and output

Filter design is harder

- Matlab's *fminsearch()* allows you to design pretty good filters even if you know nothing about filter design
- Other methods exist and are covered in Analog Electronis and Signals & Systems