Phasors

Objective:

- Find the forced response to a differential equation with a sinusoidal input using phasor analysis
- Solve AC circuits using techniques from EE206 using phasor techniques

Phasor Analysis:

The goal of phasor analysis is to use all the techniques from EE206 for DC analysis with AC circuits which include inductors and capacitors. To do this, we need to use complex numbers and phasors. With this tool, all previous techniques, like current loops, voltage notes, etc work. The only catch is your coefficients are complex numbers.

Assume all signals are in the form of e^{st} in general, or $e^{j\omega t}$ for the special case of sinusoidal inputs. The impedance of a resistor is

$$v = R \cdot i$$

The impedance of a capacitor is

$$v = \frac{1}{C}\int (i)dt$$

If i(t) is in the form of $e^{j\omega t}$, then integration becomes division by $j\omega$

$$i(t) = e^{j\omega t}$$

$$\int (i)dt = \frac{1}{j\omega}e^{j\omega} = \frac{1}{j\omega} \cdot i(t)$$

Then

$$V = \left(\frac{1}{j\omega C}\right) I$$

The impedance of a capacitor is $\frac{1}{j\omega C}$

The impedance of an inductor is

$$v = L\frac{di}{dt}$$

Assuming

$$i(t) = e^{j\omega t}$$

then

$$V = (j\omega L)I$$

The impedance of an inductor is 'jwL'.

In summary:

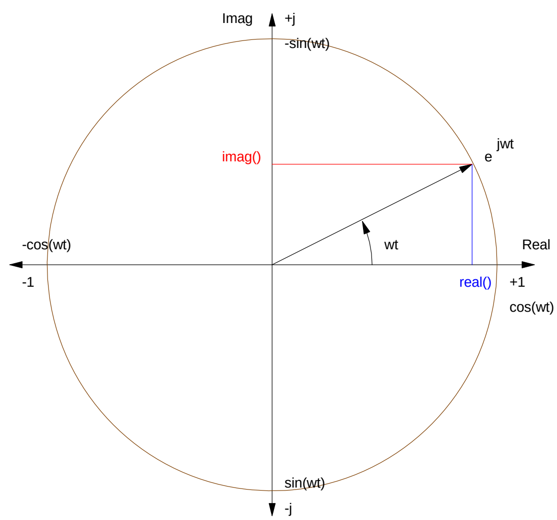
Component	Impedance
Input	e^{jwt}
R	R
L	jwL
C	1 / jwC

Sine and Cosine in Phaser Notation:

The complex exponential is

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

A graphical representation of this is shown below where the angle ωt spins counter clockwise with time at a rate of ω rad/sec.



Note the following:

If you start out at +1 and take the real part, the output starts at +1, goes to zero at 90 degrees, goes to -1, back to 0, +1, etc. It maps out a cosine wave in other words:

$$real(e^{j\omega t}) = \cos(\omega t)$$

If you start out at -j and take the real part, the output starts at 0, goes to +1, 0, -1, 0, etc. It maps out a sine wave in other words:

$$real(-j \cdot e^{j\omega t}) = \sin(\omega t)$$

In phasor notation, the $e^{j\omega t}$ is understood. The phasor notation for sine and cosine is then:

$$\cos(\omega t) \Rightarrow +1 = 1\angle 0^0$$

$$\sin(\omega t) \Rightarrow -j = 1\angle -90^0$$

This notation works both ways. When setting up a problem the input is converted to phasor notation. For example,

$$x(t) = 4\cos(3t) + 5\sin(3t)$$

is expressed as

$$X = 4 - j5$$

with the cos() understood. Similarly, when you finish phasor analysis and wind up with an answer such as

$$Y = 3 + j7$$

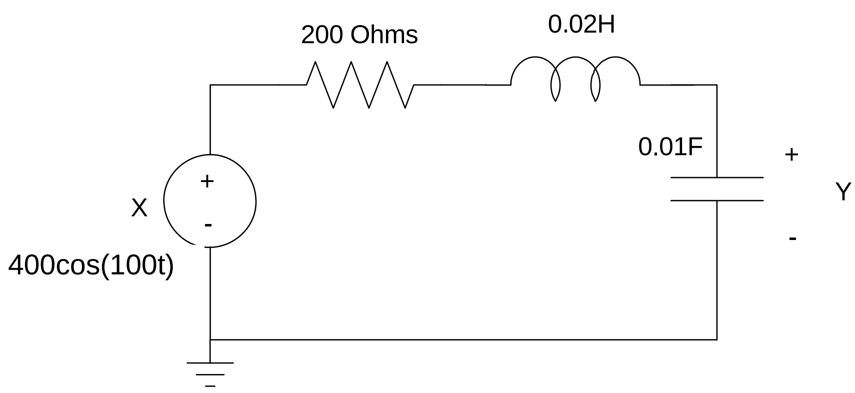
this is shorthand for

$$y(t) = 3\cos(\omega t) - 7\sin(\omega t)$$

where ω is the frequency of the output (equal to the frequency of the input or forcing function.)

Circuit Analysis with Phasors:

Assume you want to find the voltage, y(t):



First, convert to phaser notation. For this forcing function, x(t),

$$s = j100$$

Change the input to phaser notation:

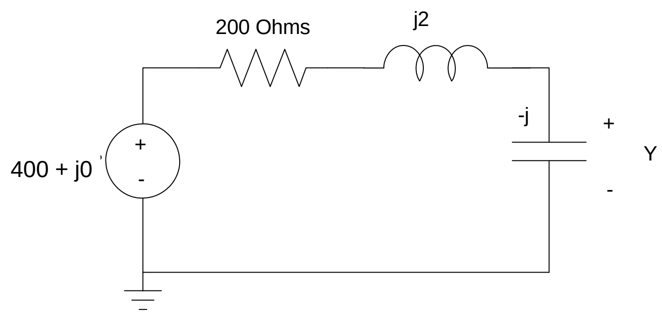
$$X = 400 + j0$$

(4 cosine + 0 sine). Change RLC to phaser impedances:

$$R \rightarrow R = 200\Omega$$

$$L \rightarrow Ls = j2\Omega$$

$$C \rightarrow \frac{1}{Cs} = -j1\Omega$$



Now, use techniques from EE206 to find Y.

You could write the voltage node equations

$$\left(\frac{Y-400}{200+j2}\right)+\left(\frac{Y}{-j}\right)=0$$

or use current division

$$Y=\left(\frac{-j}{200+2j-j}\right)\cdot(400+j0)$$

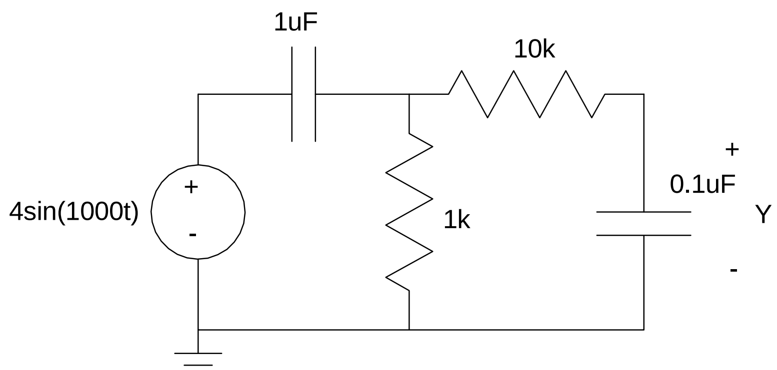
Either way, you get

$$Y=-0.01-j2.00$$

meaning

$$y(t)=-0.01\cos(100t)+2.00\sin(100t)$$

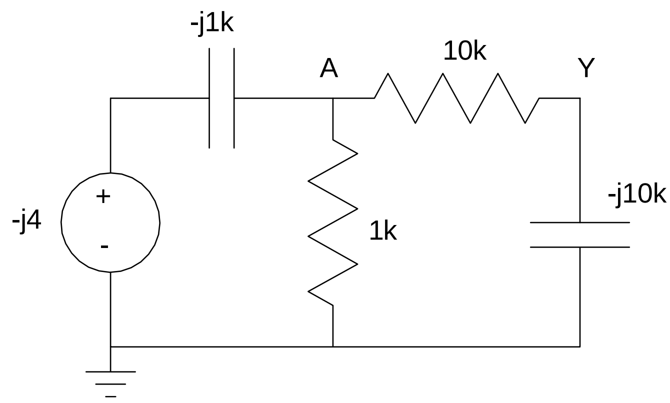
Example 2: Find the voltage at Y:



First, convert to phaser notation. In this case, due to the input being a 1000 rad/sec sine wave:

$$s = j1000$$

The phaser impedances are then:



Current loops or voltage nodes work for this circuit. Selecting voltage nodes,

$$\left(\frac{A-(-j4)}{-j1000}\right) + \left(\frac{A-0}{1000}\right) + \left(\frac{A-Y}{10000}\right) = 0$$

$$\left(\frac{Y-A}{10000}\right) + \left(\frac{Y-0}{-j10000}\right) = 0$$

Scaling both equations by 10000 to make the numbers nicer and grouping terms:

$$(11 + j10)A + (1)Y = 40$$

$$(-1)A + (1 + j)Y = 0$$

This gives two equations for two unknowns. Solving for Y (SciLab or MATLAB notation):

```
M = [11+j*10,1;-1,1+j]

    11. + 10.i    1.
  - 1.            1. + i

inv(M)*[40;0]

Va    2.0674157 - 1.7078652i
Vy    0.1797753 - 1.8876404i
```

so

$$Y = 0.17977 - j1.8876$$

or

$$y(t) = 0.17977\cos(1000t) + 1.8876\sin(1000t)$$

If you prefer polar notation:

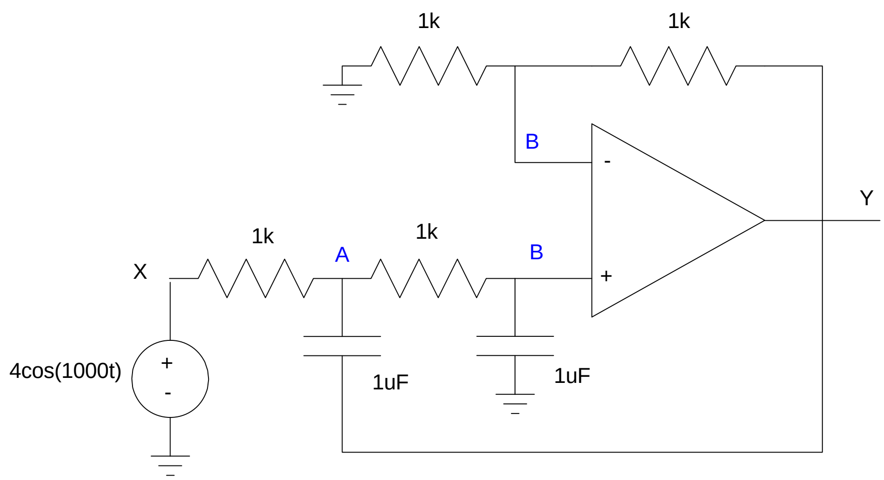
$$Y = 1.896\angle -84.56^0$$

or

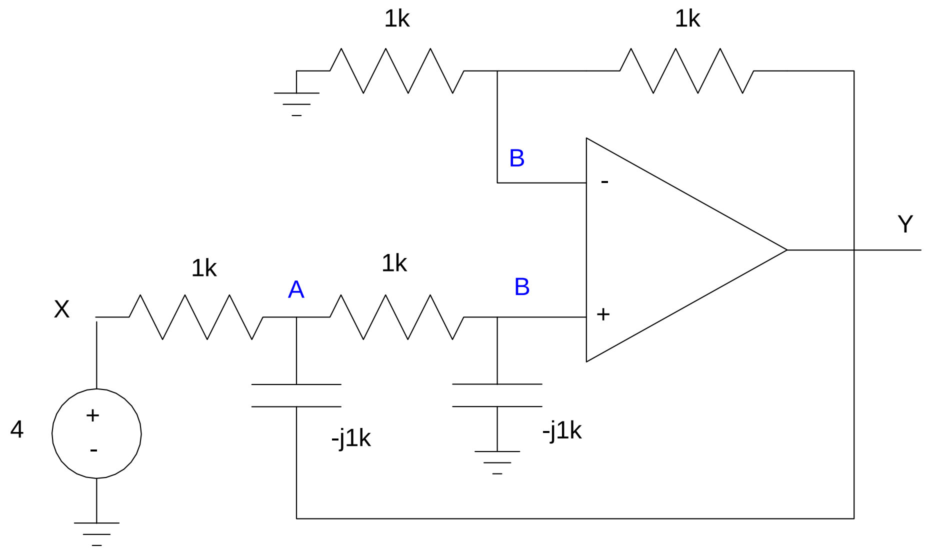
$$y(t) = 1.896\cos(1000t - 84.56^0)$$

Example 3: Op Amp Circuit:

Find y(t):



First. convert to phaser notation. In this case, $s = j1000$



Next, write the voltage node equations:

A $\left(\frac{A-4}{1000}\right)+\left(\frac{A-Y}{-j1000}\right)+\left(\frac{A-B}{1000}\right)=0$

V- $\left(\frac{B-A}{1000}\right)+\left(\frac{B-0}{-j1000}\right)=0$

V+ $\left(\frac{B-0}{1000}\right)+\left(\frac{B-Y}{1000}\right)=0$

Note that you can't write a voltage node equation at Y. The op-amp puts out whatever current it takes to force V- = V+. Since you don't know this current, you can't sum the currents to zero. The fourth equation already incorporated in the above three is

$$V^+=V^-$$

To make the numbers nicer, lets scale all three equations by 1000 and group terms:

A $(2+j)A+(-j)Y+(-1)B=4$

V- $(-1)A+(0)Y+(1+j)B=0$

V+ $(0)A+(-1)Y+(2)B=0$

Solving three equations for three unknowns (again using MATLAB):

```
-->W = [2+j,-j,-1;-1,0,1+j;0,-1,2]
 W  =
```

```
     2. + i    - i    - 1.
   - 1.          0      1. + i
     0         - 1.     2.


-->inv(W)*[4;0;0]
 ans  =

Va    4. - 4.i
Vy  - 8.i
Vb  - 4.i
```

The way I set up the problem, Y was the second column in the 3x3 matrix. Likewise, Y will be the second entry in the answer:

$$Y = -j8$$

and

$$y(t) = 8\sin(1000t)$$