Natural Response of RL & RC Circuits

Phasor analysis looks at the stead-state repsonse of RLC circuits. The transient response is a bit more difficult to analyze. The basic equations for an inductor and capacitor are:

$$v = L\frac{di}{dt}$$

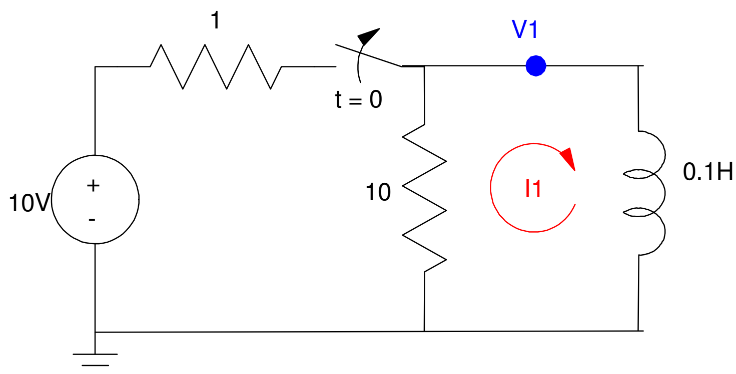
and

$$i = C\frac{dv}{dt}$$

This means that you need to solve a differential equation when analyzing a circuit with inductors and capacitors. Hence, the reason you took Calculus I, Calculus II, Calculus III, and Differential Equations.

Natural Response of an RL Circuit

Consider the fosllowing RL circuit where the switch opens up at t=0



Writing the loop equation around IL gives

$$L\frac{di}{dt} + iR = 0$$

$$0.1\frac{di}{dt} + 10i = 0$$

and an intial condition (just prior to t = 0)

$$i(0) = \left(\frac{10V}{1\Omega}\right) = 10A$$

One way to solve this type of equation is to 'guess' the answer. The differential equation is

$$\frac{di}{dt} + 100i = 0$$

'Guess' the solution is of the form

$$i(t) = a \cdot e^{pt}$$

As a check, plug this into the previous differential equation

$$\frac{di}{dt} + 100i = 0$$

$$\frac{d}{dt}(a \cdot e^{pt}) + 100(a \cdot e^{pt}) = 0$$

$$(p) \cdot a \cdot e^{pt} + 100 \cdot a \cdot e^{pt} = 0$$

$$(p + 100) \cdot ae^{pt} = 0$$

Either

a = 0 (trivial solution)

or

$$p = -100$$

meaning

$$i(t) = a \cdot e^{-100t}$$

Plugging in the initial condition

$$i(0) = 1A$$

tells you that a = 1

$$i(t = 0) = 10 = a \cdot e^{0}$$

$$a = 10$$

Hence

$$\boxed{i(t) = 10e^{-100t}} \qquad t > 0$$

The voltage, V1, is

$$v = L\frac{di}{dt}$$

$$v_1 = 0.1 \cdot \frac{d}{dt}(10e^{-100t})$$

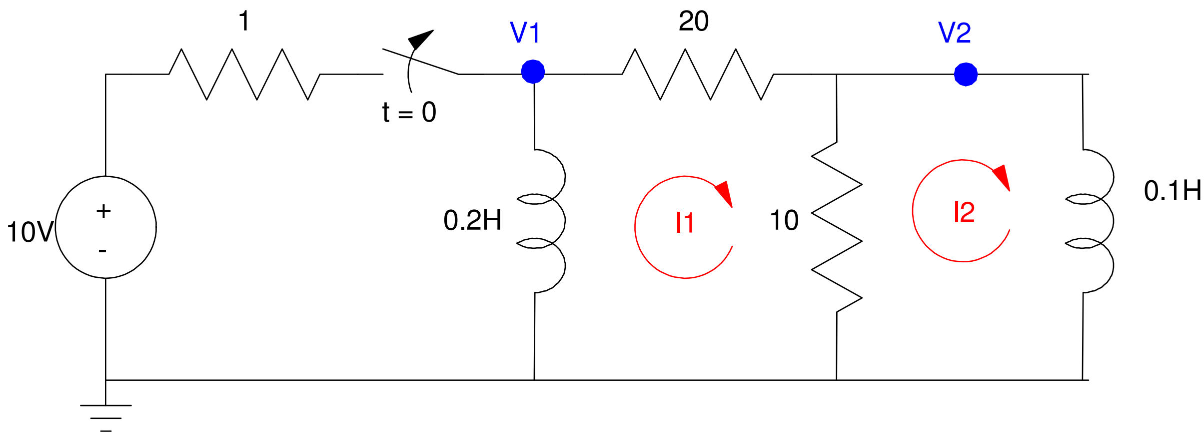
$$v_1 = 0.1 \cdot (-1000 \cdot e^{-100t})$$

$$\boxed{v_1 = -100 \cdot e^{-100t}} \qquad t > 0$$

Note that a 10V source produced 100V across the inductor when the switch was opened. This is how spark plugs work: energize an inducotor (the alternator) then open up a switch. The energy in the magnetic field collapses and finds a path to ground. In this case, the path goes through the spark plug.

Natural Response of a 2-stage RL Filter

Find the current, i2(t). Assume the switch has been closde for a long time at opens at t = 0.



The initial conditions are

$$i_1(0) = \frac{10V}{1\Omega} = 10A$$

$$i_2(0) = 0$$

For this circuit, you'll get a 2nd-order differential equation since there are two energy storage elements. Writing the loop equations:

$$0.2\frac{di_1}{dt} + 20i_1 + 10(i_1 - i_2) = 0$$

$$10(i_2 - i_1) + 0.1\frac{di_2}{dt} = 0$$

Rewriting these

$$\frac{di_1}{dt} + 150i_1 = 50i_2$$

$$100i_1 = \frac{di_2}{dt} + 100i_2$$

Substitute the second equation into the first:

$$\frac{d}{dt}\left(0.01\frac{di_2}{dt} + i_2\right) + 150\left(0.01\frac{di_2}{dt} + i_2\right) = 50i_2$$

$$0.01\frac{d^2i_2}{dt^2} + 2.5\frac{di_2}{dt} + 100i_2 = 0$$

$$\frac{d^2i_2}{dt^2} + 250\frac{di_2}{dt} + 10,000i_2 = 0$$

Assume

$$i_2(t) = ae^{pt}$$

then

$$p^2 \cdot ae^{pt} + 250p \cdot ae^{pt} + 10,000 \cdot ae^{pt} = 0$$

$$(p^2 + 250p + 10,000) \cdot ae^{pt} = 0$$

$$(p+50)(p+200) \cdot ae^{pt} = 0$$

Either a = 0 (trivial solution) or p = {-50, -200}

$$i_2(t) = a \cdot e^{-50t} + b \cdot e^{-200t}$$

Plug in the initial conditions

$$i_2(0) = 0 = a + b$$

$$i_1(t) = 0.01\frac{di_2}{dt} + i_2$$

$$i_1(0) = 10 = 0.01(-50a - 200b) + (a+b)$$

Solving 2 equtions for 2 unknowns

$$a + b = 0$$

$$0.5a - b = 10$$

$$1.5a = 10$$

$$a = 6.667$$

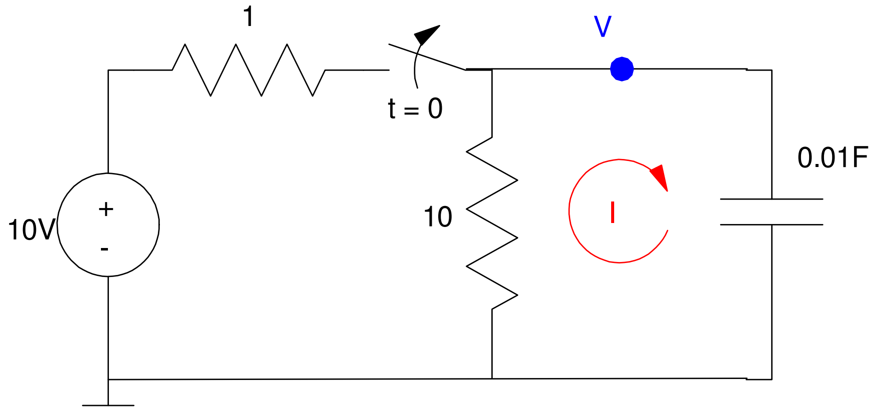
$$b = -6.667$$

resulting in

$$\boxed{i_2(t) = 6.667e^{-50t} - 6.667e^{-200t}} \qquad t > 0$$

Natural Response of a RC Circuit:

Assume for the following circuit the switch opens at t=0. Find V(t)



The initial conidtion is

$$v_1(0) = \left(\frac{10}{10+1}\right)10V = 9.09V$$

The differential equation for this circuit comes from the voltage node equation:

$$\left(\frac{v_1}{R}\right) + C\frac{dv_1}{dt} = 0$$

$$0.1v_1 + 0.01\frac{dv_1}{dt} = 0$$

$$\frac{dv_1}{dt} + 10v_1 = 0$$

Guess V is of the form

$$v_1(t) = a \cdot e^{pt}$$

Substituting into the differential equation

$$\frac{dv_1}{dt} + 10v_1 = 0$$

$$\frac{d}{dt}(a \cdot e^{pt}) + 10(a \cdot e^{pt}) = 0$$

$$(ap \cdot e^{pt}) + 10(a \cdot e^{pt}) = 0$$

$$(p+10) \cdot (ae^{pt}) = 0$$

Either a = 0 (trivial solution) or p = -10. This means

$$v_1(t) = a \cdot e^{-10t}$$

Plugging in the initial condition

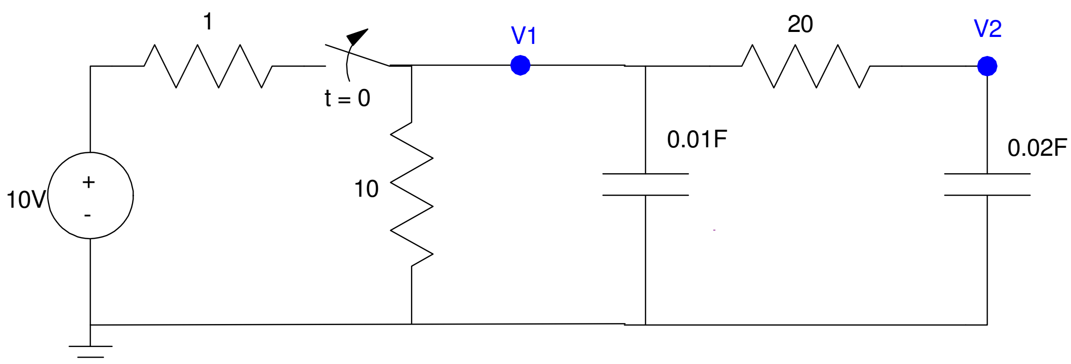
$$v_1(0) = 9.09 = a$$

results in

$$\boxed{v_1(t) = 9.09e^{-10t}} \qquad t > 0$$

2-Stage RC Filter

Find v2(t) for the following RC circuit. Assume the switch was closd for a long time before t = 0 and opened up at t=0.



The initial conditions are

$$v_1(0) = \left(\frac{10}{10+1}\right) 10V = 9.09V$$

$$v_2(0) = v_1(0) = 9.09V$$

The differential equations which describe this circuit come form the voltage node equtions:

$$\left(\frac{v_1}{10}\right) + 0.01\frac{dv_1}{dt} + \left(\frac{v_1 - v_2}{20}\right) = 0$$

$$\left(\frac{v_2 - v_1}{20}\right) + 0.02\frac{dv_2}{dt} = 0$$

Rewriting this

$$\frac{dv_1}{dt} + 15v_1 = 5v_2$$

$$\frac{dv_2}{dt} + 2.5v_2 = 2.5v_1$$

Substutute for v1 to find v2:

$$\frac{d}{dt}\left(0.4\frac{dv_2}{dt} + v_2\right) + 15\left(0.4\frac{dv_2}{dt} + v_2\right) = 5v_2$$

$$\frac{d^2v_2}{dt^2} + 17.5v_2 + 25v_2 = 0$$

Assume

$$v_2(t) = a \cdot e^{pt}$$

then

$$p^2 a \cdot e^{pt} + 17.5pa \cdot e^{pt} + 37.5a \cdot e^{pt} = 0$$

$$(p^2 + 17.5p + 25)a \cdot e^{pt} = 0$$

$$(p + 1.569)(p + 15.931) \cdot ae^{pt} = 0$$

meaning

p = {-1.569, -15.931}

and

$$v_2(t) = ae^{-1.569t} + be^{-15.931t}$$

Plugging in the initial conditions

$$v_2(0) = 9.09 = a + b$$

and

$$v_1(t) = 0.4\frac{dv_2}{dt} + v_2$$

$$v_1(0) = 9.09 = 0.4(-1.569a - 15.931b) + (a + b)$$

Solving 2 equations for 2 unknowns results in

$$a = 10.083$$

$$b = -0.993$$

and

$$\boxed{v_2(t) = 10.083e^{-1.569} - 0.993e^{-15.931t}} \qquad t > 0$$