Capacitors &
The Heat Equation
EE 206 Circuits I
Jake Glower - Lecture #17

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

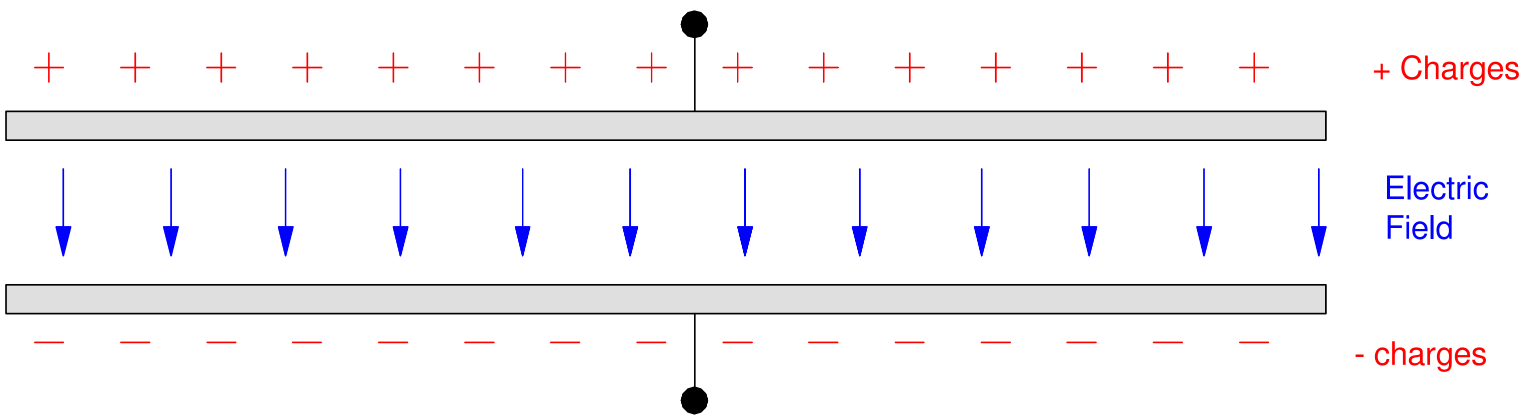
Capacitors

A capacitor is a set of parallel plates[1] with the capacitance equal to

$$C = \varepsilon \frac{A}{d} \quad \text{(Farads)}$$

where

- ε is the dielectric constant of the material between plates (air = $8.84 \cdot 10^{-12}$)
- A is the area of the capacitor, and
- d is the distance between plates.



[1] http://www.electronics-tutorials.ws/

The area you need for 1 Farad with plates 1mm apart is

$$1 = (8.84 \cdot 10^{-12})\frac{A}{0.001m}$$

$$A = 113,122,171m^2$$

Equal to 10.6km x 10.6km

- Most capacitors on the order of μF or nF

The charge stored is

$$Q = C \cdot V$$

Q = charge (Coulombs - one Coulomb is equal to $6.242 \cdot 10^{18}$ electrons).

Voltage - Current Relationship

$$Q = CV$$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$$

Assuming the capacitance is constant

$$\boxed{I = C\frac{dV}{dt}}$$

This means that capacitors are integrators:

$$V = \frac{1}{C}\int I \cdot dt$$

Capacitors and Energy Storage

The energy stored is

- $E = \frac{1}{2}CV^2$
- 12.5J = 12.5kW for 1ms
- Capacitors provide energy for short bursts

Item	Energy (Joules)	Cost	$ / MJ
1 pound Wyoming Coal	3,600,000	$0.028	$0.0078
1 pound ND Lignite	1,565,217	$0.017	$0.0108
1 pint of gasoline	15,000,000	$0.37	$0.0247
Lithium battery (D cell)	246,240	$22	$89.43
1F Capacitor (5V)	12.5	$2.87	$229,600

Numerical Integration and Capacitors

Capacitors are inherently integrators:

$$I = C\,\frac{dV}{dt}$$

$$V = \frac{1}{C}\int I\,dt$$

Differential equations are required to described circuits with capacitors

- Each capacitor adds a 1st-order differential equation
- N capacitors means an Nth-order differential equation

Calculus: Solving differtial equations using calculus

Circuits I: Solving differetial equations using phasors (coming soon) and numerical methods

Circuits II, Signals & Systems: Solving differential equtions using LaPlace transforms

Numerical Integration

- Solve a differential equation using numerical methods (i.e. Matlab)
- Whole field of mathematics deals with numerical integration

Several types of numerical integration

- Euler Integration
- Trapezoid Rule
- Runge Kutta Integration
- more...

All are approximate

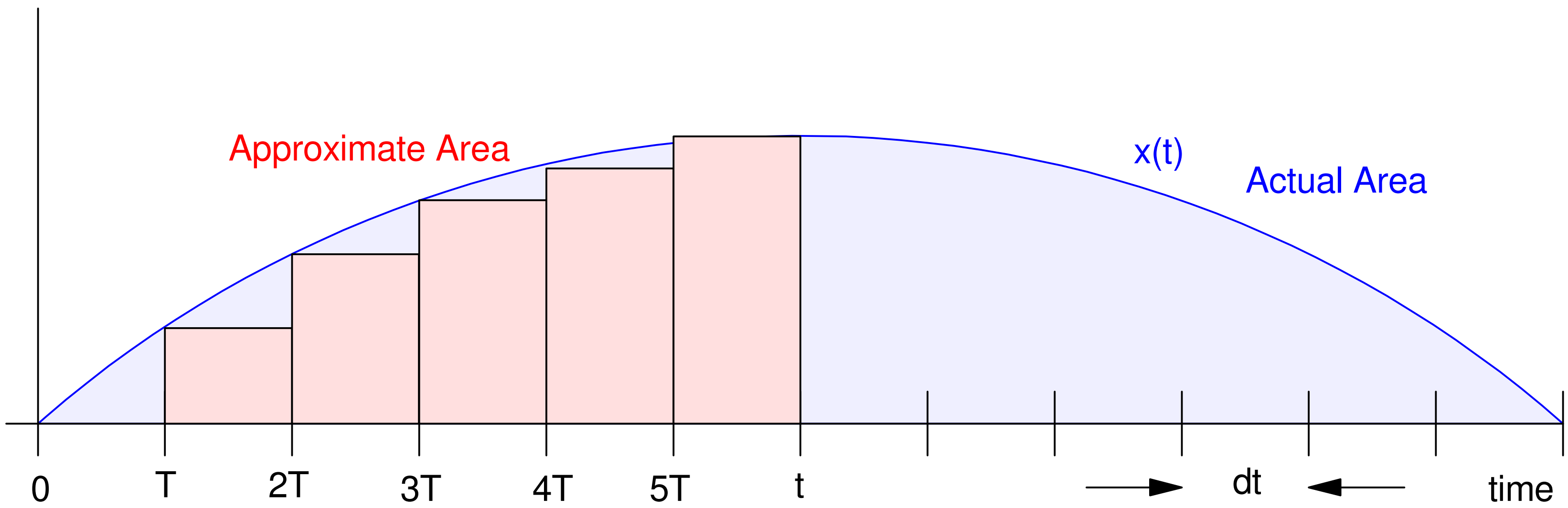
- Use Calculus or LaPlace transforms to get closed-form, exact solutions

Euler Integration

- $y(t) = y(t-T) + x(t) \cdot dt$
- Sample x(t) every T seconds,
- Use rectangles to approximate the area every T seconds, and
- Sum up the area of each rectangle.

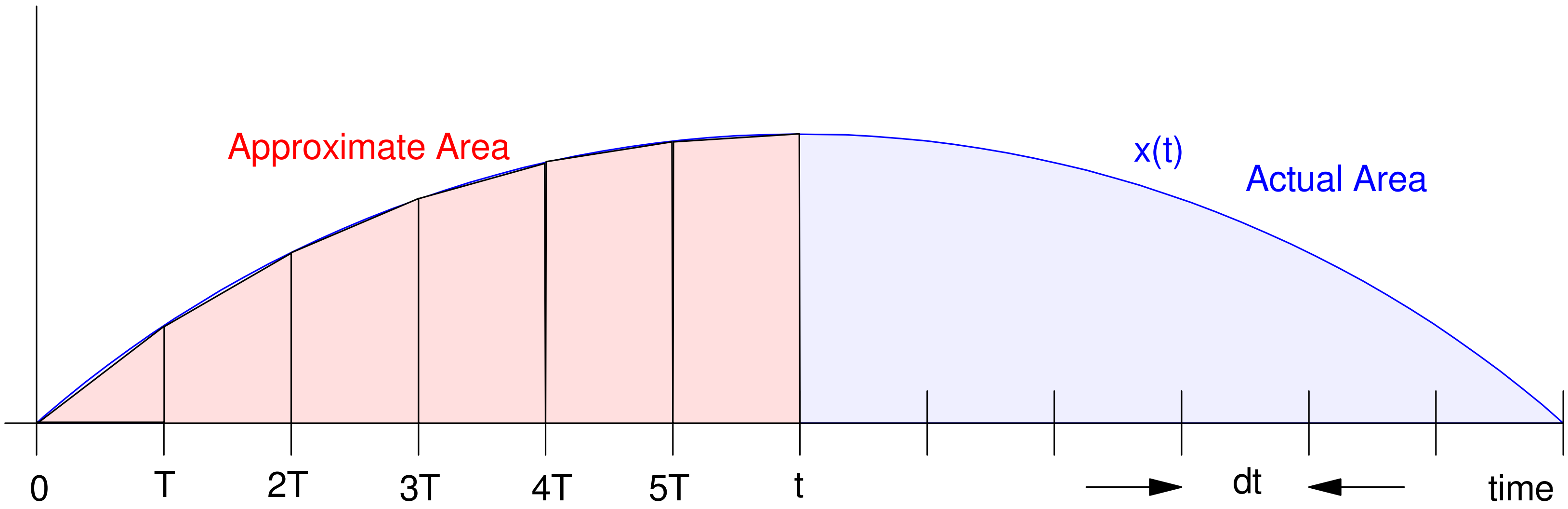
Result is the simplest and least accurate of the three forms.

- Not too bad if you keep the sampling time (dt) small.



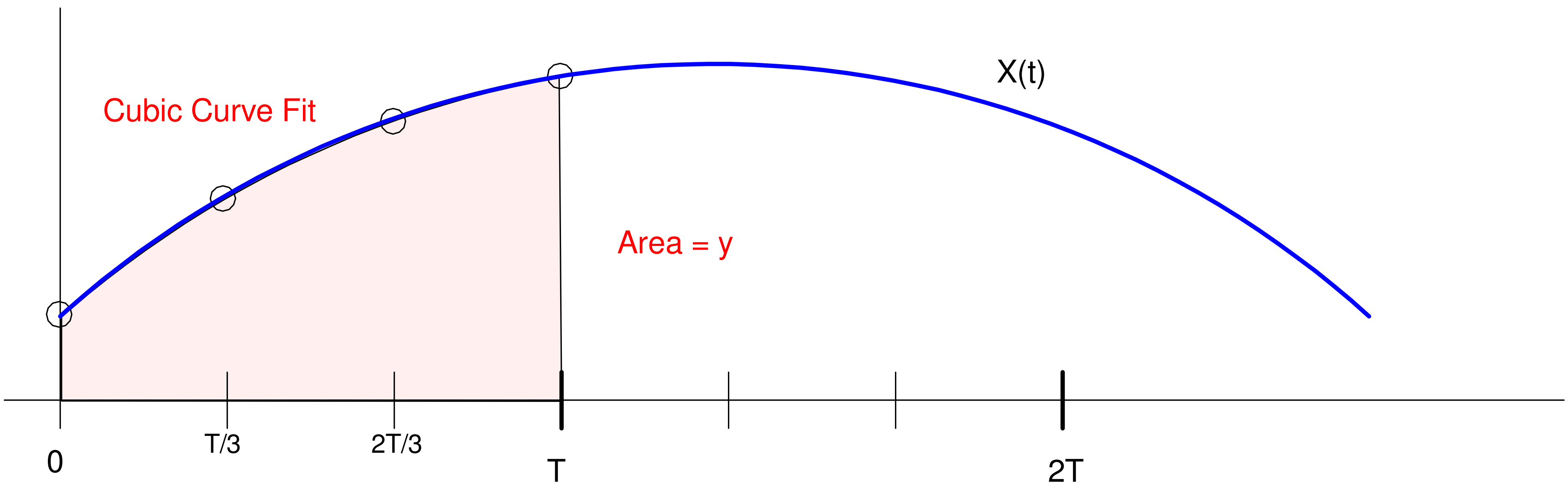
Trapezoid (Bilinear) Integration:

- $y(t) = y(t-T) + \left(\frac{x(t)+x(t-T)}{2}\right) \cdot dt$
- Sample x(t) every T seconds,
- Use trapezoids to approximate the area every T seconds
- *Much* better than Euler
- Required memory (need to recall previous input)



Runge Kutta Integration:

- Sample x(t) every T seconds,
- Use parabola, cubics, etc. to approximate the area
- Required memory and data inbetween samples



Stick with Euler integration

To find the voltage across a capacitor

- Compute the current to the capacitor, and
- Integrate using Euler integration:

```
dV = I / C
V = V + dV * dt
```

Example 1: 1-Stage RC Circuit

Find V1(t) with

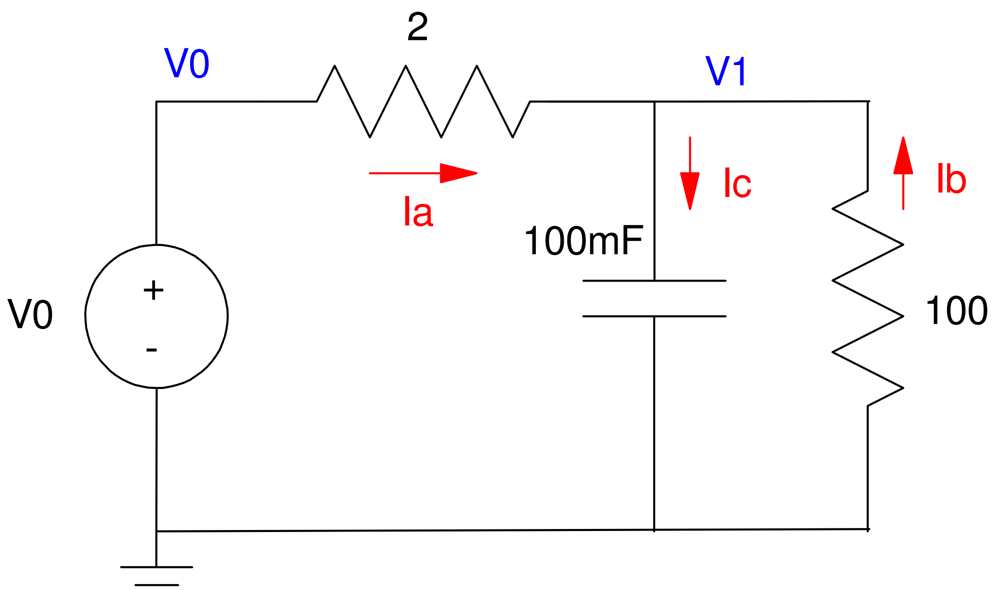
$$V_0(t) = 10u(t) = \begin{cases} 0V & t < 0 \\ 10V & t > 0 \end{cases}$$

Solution

$$I_c = I_a + I_b$$

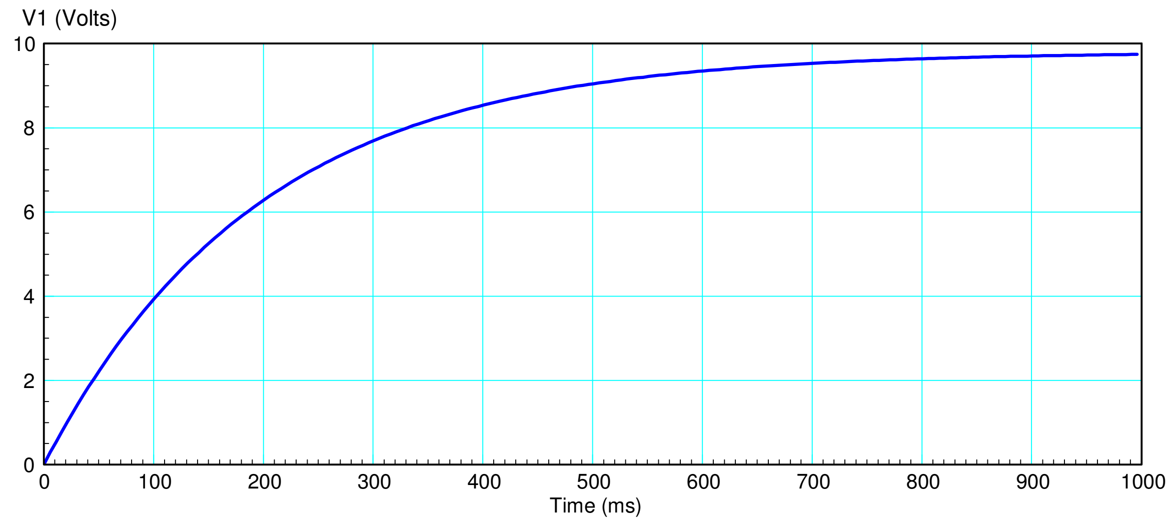
$$C\frac{dV_1}{dt} = I_c = \left(\frac{V_0 - V_1}{2}\right) + \left(\frac{0 - V_1}{100}\right)$$

$$\frac{dV_1}{dt} = -5.1\,V_1 + 5\,V_0$$



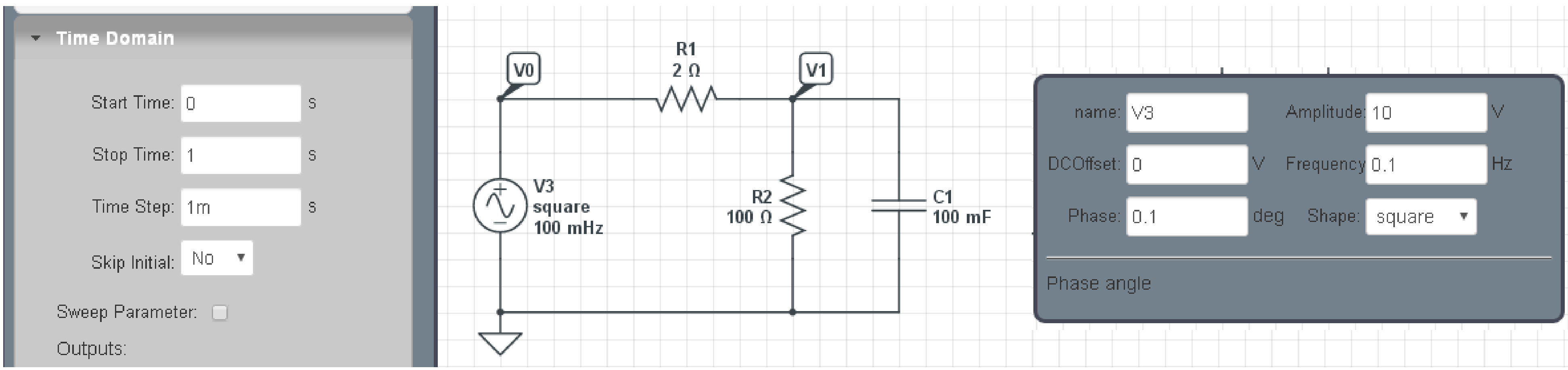
Solve in Matlab

```
% 1-stage RC Filter

V = 0;
V0 = 10;
dt = 0.01;
t = 0;
Y = [];

while(t < 1)
   dV  = -5.1*V + 5*V0;
   V = V +  dV*dt;
   t = t + dt;
   Y = [Y ; V];
end

t = [1:length(Y)]' * dt;

plot(t, Y);
```



Solve in CircuitLab

- V1 = 10V square wave, 0.1Hz, 0.1 degree phase shift
- Run a time-domain simulation for 1 second
- Gives the same answer
- CircuitLab also solves using numerical integration



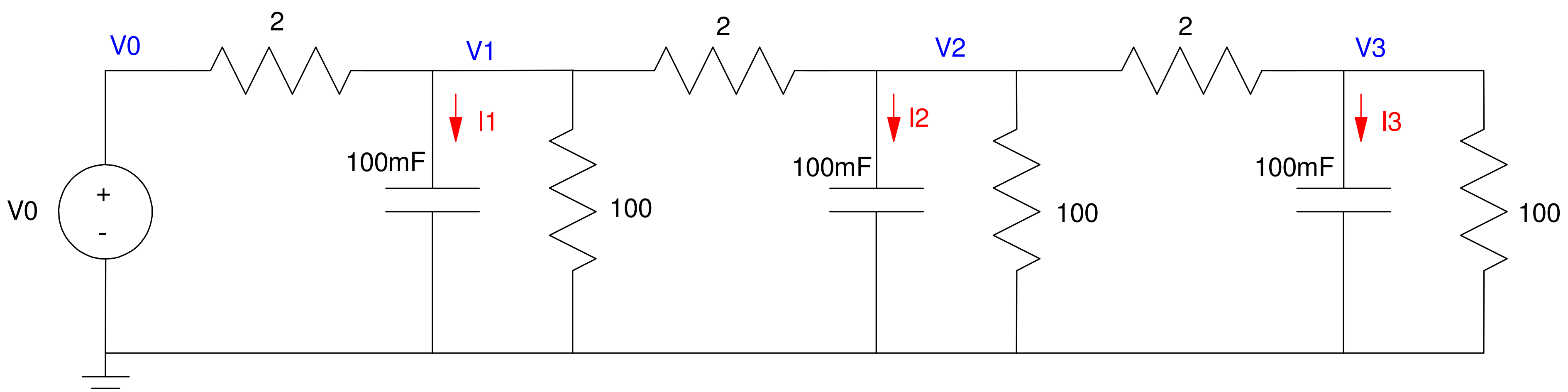
Example 2: 3-Stage RC Filter

$$V_0(t) = 10u(t)$$

$$C_1\frac{dV_1}{dt} = I_1 = \left(\frac{V_0-V_1}{2}\right) + \left(\frac{0-V_1}{100}\right) + \left(\frac{V_2-V_1}{2}\right)$$

$$C_2\frac{dV_2}{dt} = I_2 = \left(\frac{V_1-V_2}{2}\right) + \left(\frac{0-V_2}{100}\right) + \left(\frac{V_3-V_2}{2}\right)$$

$$C_3\frac{dV_3}{dt} = I_3 = \left(\frac{V_2-V_3}{2}\right) + \left(\frac{0-V_3}{100}\right)$$

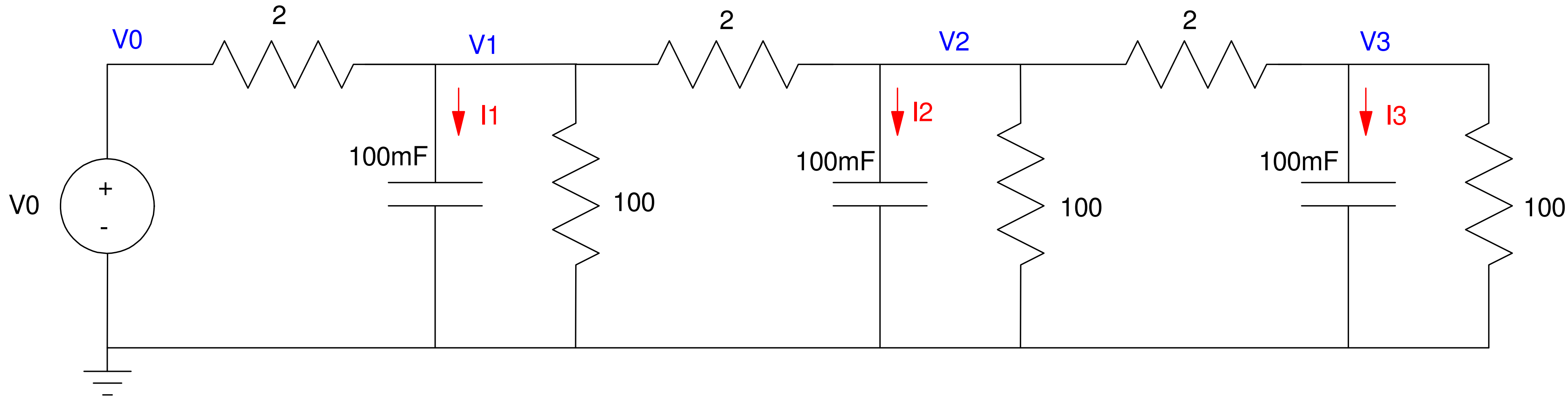


Next, determine dV/dt

$$\frac{dV_1}{dt} = 5V_0 - 10.1V_1 + 5V_2$$

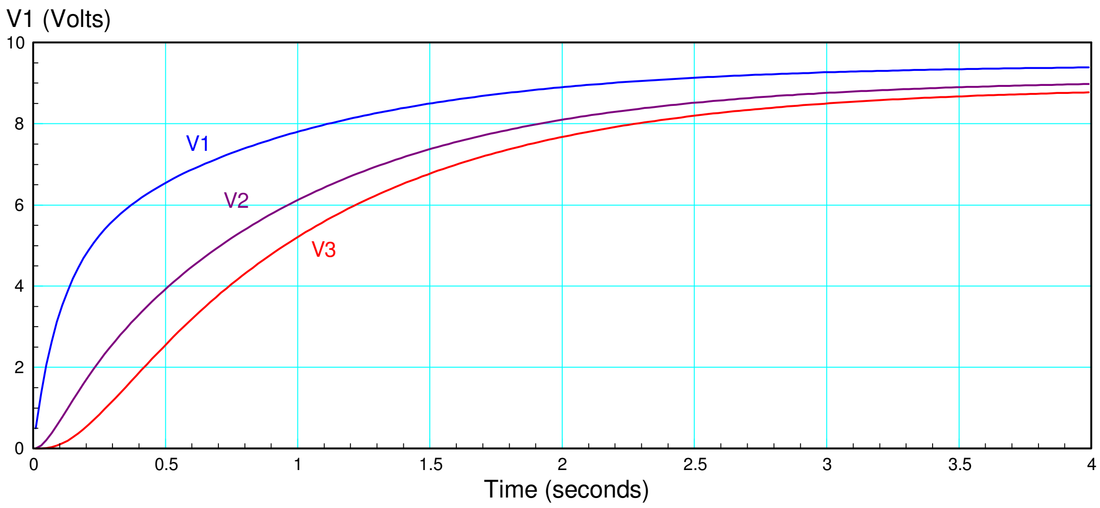
$$\frac{dV_2}{dt} = 5V_1 - 10.1V_2 + 5V_3$$

$$\frac{dV_3}{dt} = 5V_2 - 5.1V_3$$



In Matlab:

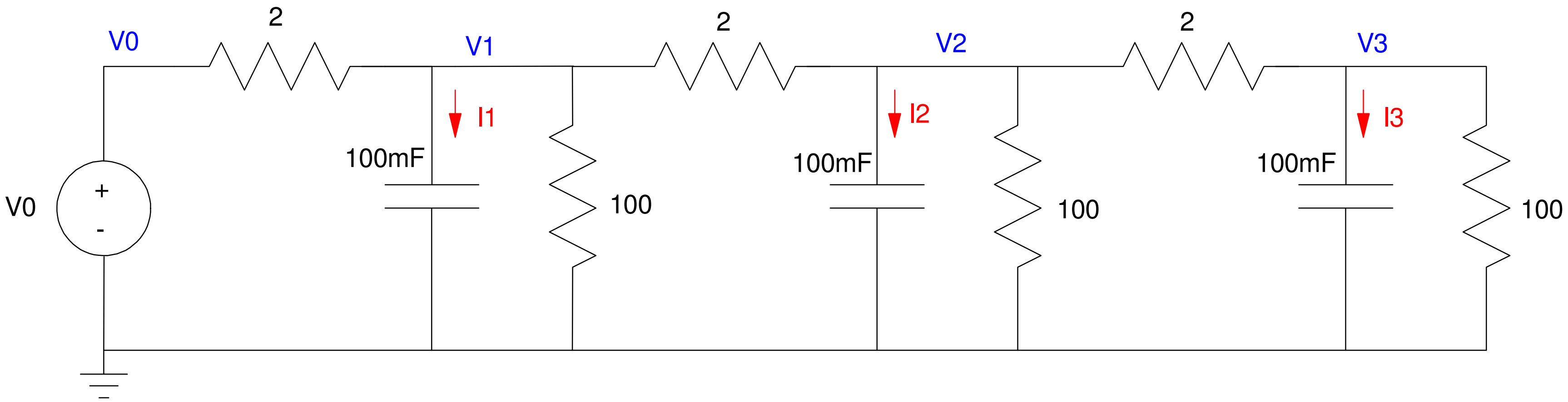
```
V0 = 10;
V1 = 0;  V2 = 0;  V3 = 0;
dt = 0.01;
t = 0;
Y = [];

while(t < 4)
   dV1  = 5*V0 -10.1*V1 + 5*V2;
   dV2  = 5*V1 -10.1*V2 + 5*V3;
   dV3  = 5*V2 -5.1*V3;

   V1 = V1 + dV1 * dt;
   V2 = V2 + dV2 * dt;
   V3 = V3 + dV3 * dt;

   t = t + dt;

   Y = [Y ; [V1, V2, V3] ];
end

t = [1:length(Y)]' * dt;
plot(t, Y);
```



Case 3: 10-Stage RC Filter: Heat Equation

$$\frac{dV_n}{dt} = 5V_{n-1} - 10.1V_n + 5V_{n+1} \qquad\qquad 1 < \mathrm{n} < 9$$

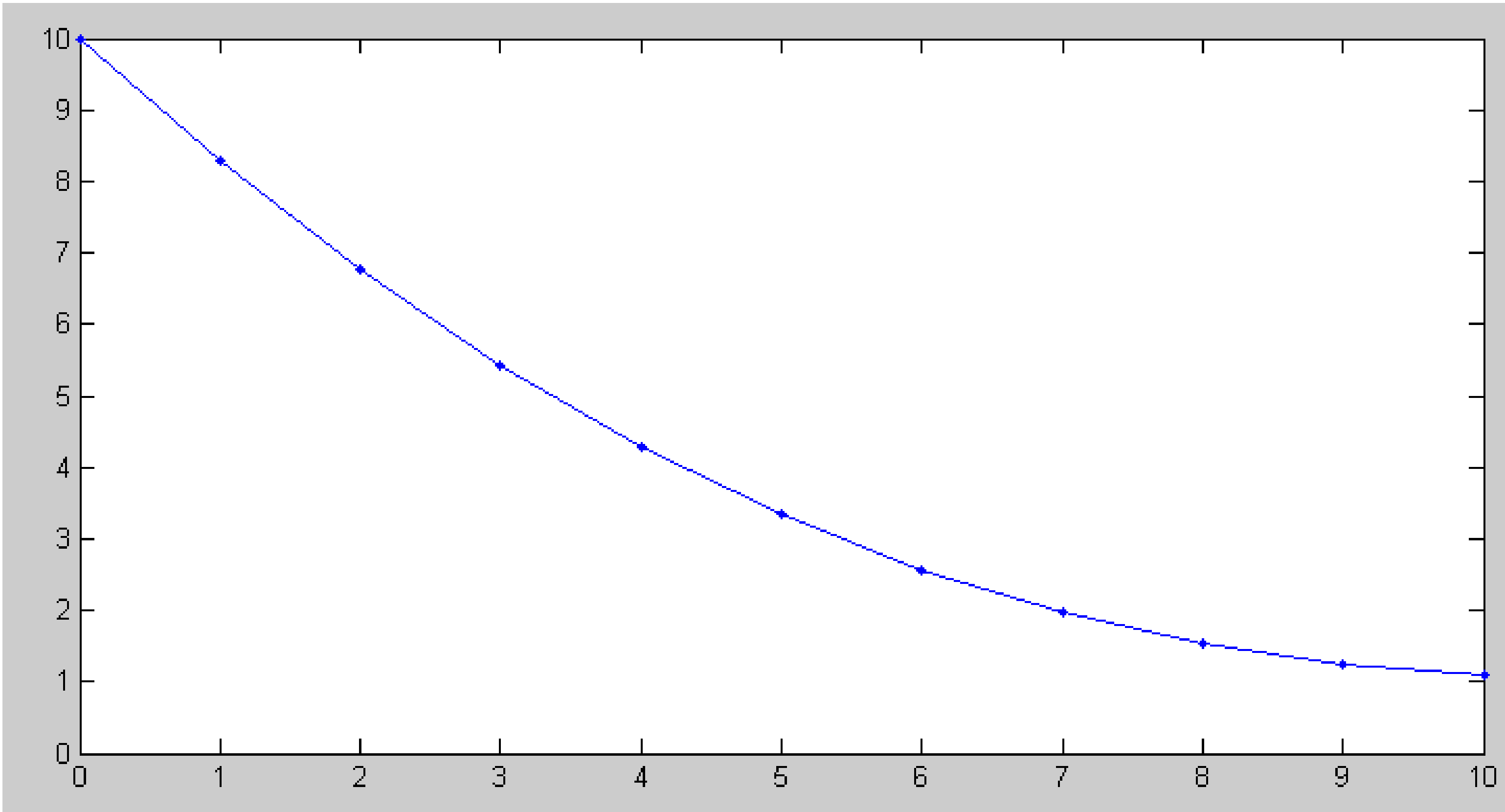
$$\frac{dV_{10}}{dt} = 5V_9 - 5.1V_{10}$$



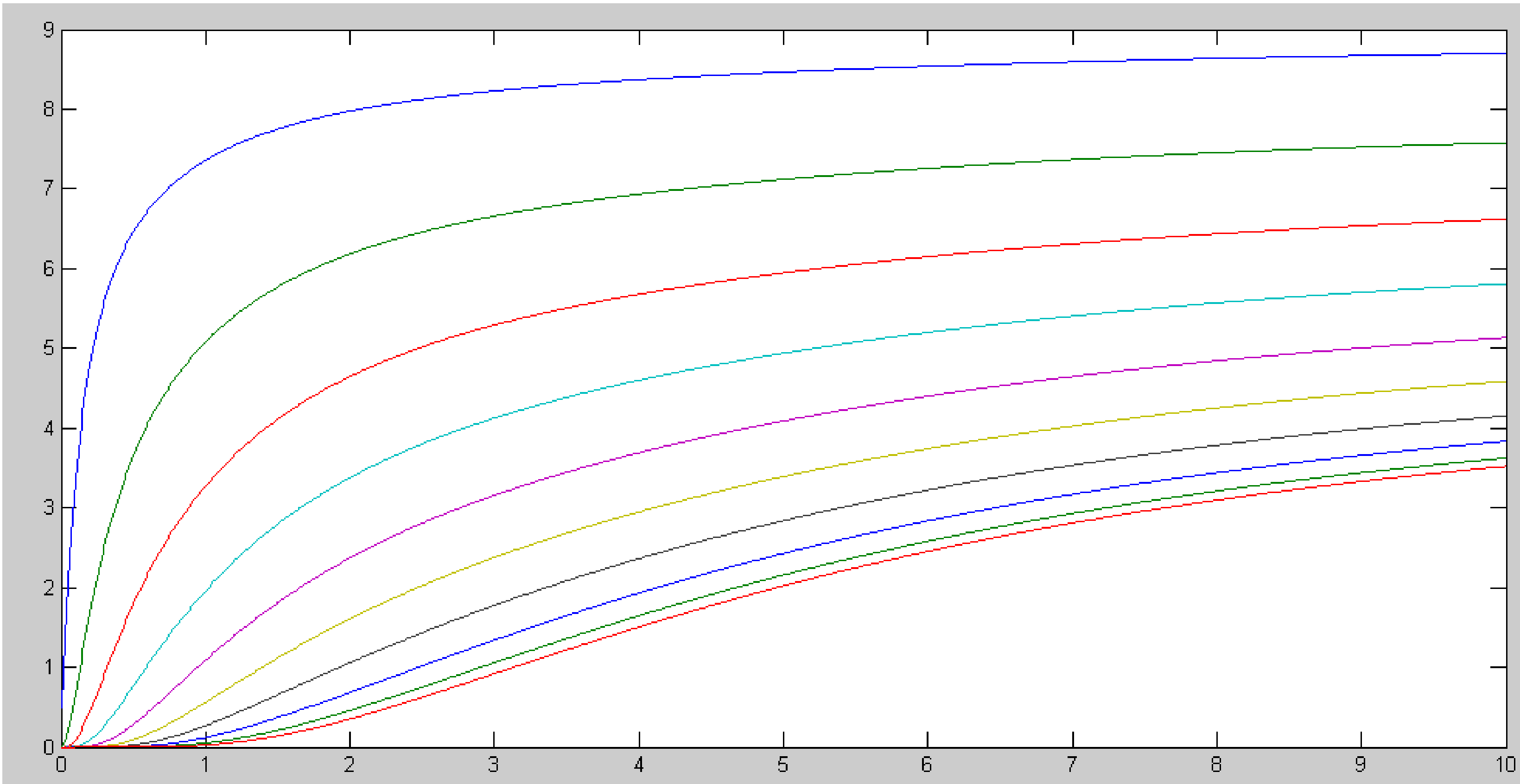
In Matlab, use a for-loop:

```
% 10-stage RC Filter

V0 = 10;
V = zeros(10,1);
dV = 0*V;
dt = 0.01;
t = 0;
Y = [];

while(t < 10)
   dV(1)  = 5*V0 -10.1*V(1) + 5*V(2);
   for i=2:9
       dV(i) = 5*V(i-1) - 10.1*V(i) + 5*V(i+1);
   end
   dV(10) = 5*V(9) - 5.1*V(10);

   V = V + dV * dt;
   t = t + dt;

   N = [0:10];
   plot(N, [V0; V], 'b.-');
   ylim([0,10]);
   pause(0.01);
```

For the first 10 seconds, this program shows the voltages along the circuit



Then, the final voltages vs. time are displayed



Note: This program

- Solves a 10-order compled differential equation
- V1..V10 represent the voltages on each capacitor as they charge
- V1..V10 also are the temperatures along a metal bar as they heat up

Coupled 1st-order differential equtions

- Describe RC circuits
- Describe heat flow
- Are called *the heat equation*

Eigenvalues and Eigenvectors

The dynamics for the 10-stage RC filter are:

$$\frac{dV_1}{dt} = \dot{V}_1 = 5V_0 - 10.1V_1 + 5V_2$$

$$\frac{dV_2}{dt} = \dot{V}_2 = 5V_1 - 10.1V_2 + 5V_3$$

$$\vdots$$

$$\frac{dV_9}{dt} = \dot{V}_9 = 5V_8 - 10.1V_9 + 5V_{10}$$

$$\frac{dV_{10}}{dt} = \dot{V}_{10} = 5V_9 - 5.1V_{10}$$

In matrix form, this can be written as

$$\dot{V} = AV + BV_0$$

or

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \dot{V}_5 \\ \dot{V}_6 \\ \dot{V}_7 \\ \dot{V}_8 \\ \dot{V}_9 \\ \dot{V}_{10} \end{bmatrix} = \begin{bmatrix} -10.1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -10.1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -10.1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10.1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -10.1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -10.1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -10.1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -10.1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -10.1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5.1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_0$$

Matrix A is a 10x10 matrix:

```
A = zeros(10,10);
for i=1:9
   A(i,i) = -10.1;
   A(i,i+1) = 5;
   A(i+1,i) = 5;
   end
A(10,10) = -5.1;
```

```
A =

  -10.1000    5.0000         0         0         0         0         0         0         0         0
    5.0000  -10.1000    5.0000         0         0         0         0         0         0         0
         0    5.0000  -10.1000    5.0000         0         0         0         0         0         0
         0         0    5.0000  -10.1000    5.0000         0         0         0         0         0
         0         0         0    5.0000  -10.1000    5.0000         0         0         0         0
         0         0         0         0    5.0000  -10.1000    5.0000         0         0         0
         0         0         0         0         0    5.0000  -10.1000    5.0000         0         0
         0         0         0         0         0         0    5.0000  -10.1000    5.0000         0
         0         0         0         0         0         0         0    5.0000  -10.1000    5.0000
         0         0         0         0         0         0         0         0    5.0000   -5.1000
```

A has 10 eigenvalues and 10 eigenvectors:

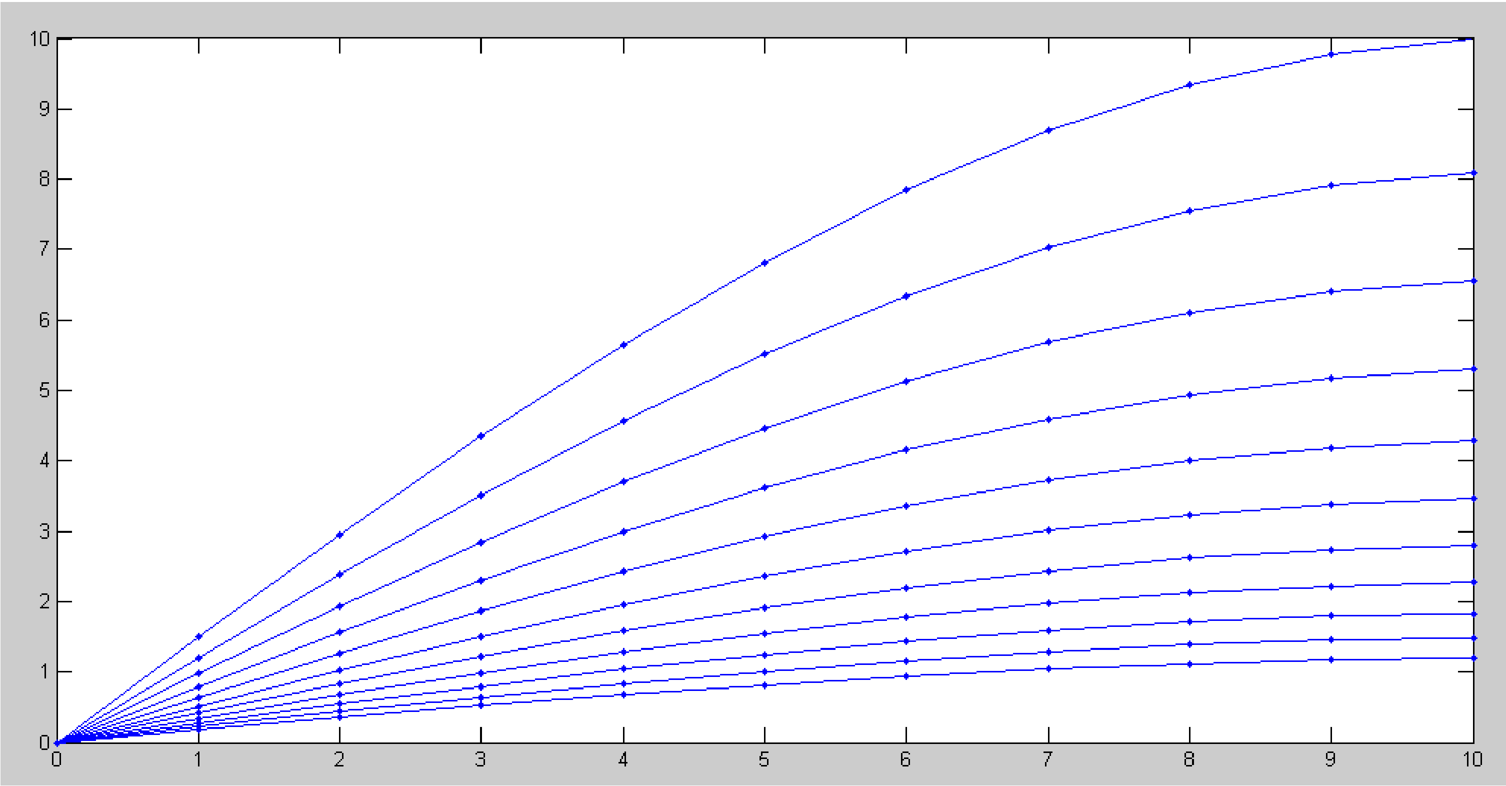
```
[M,V] = eig(A)

M = Eigenvectors:

   -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
    0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
   -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412   -0.0000    0.3412   -0.4255    0.1894
    0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
   -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
    0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255    0.0000   -0.4255   -0.1894    0.3412
   -0.3780    0.3780   -0.0000    0.3780   -0.3780    0.0000    0.3780   -0.3780   -0.0000    0.3780
    0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
   -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
    0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352

V = Eigenvalues:

  -19.6557  -18.3624  -16.3349  -13.7534  -10.8473   -7.8748   -5.1000   -2.7695   -1.0903   -0.2117
```

The eigenvalues tell you *how* the mode behaves

The eigenvector tells you *what* behaves that way.

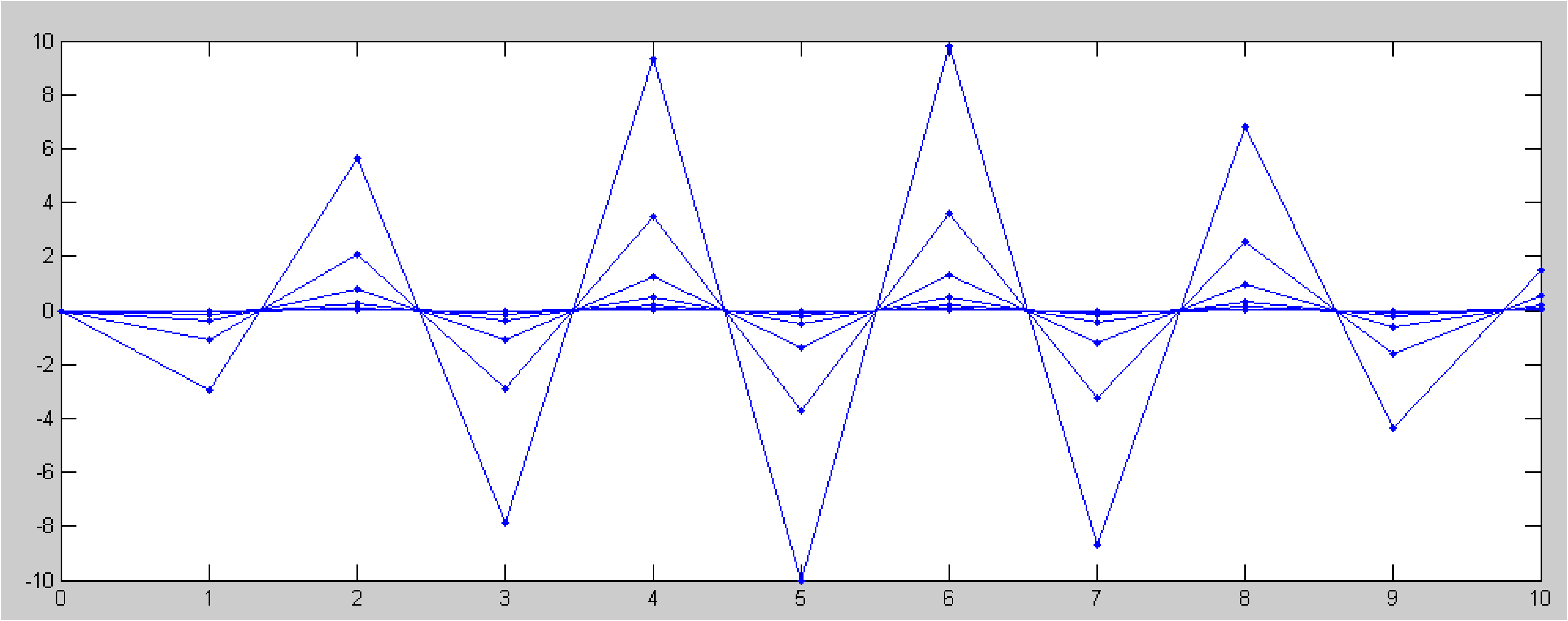
Slow Eigenvector

- V0 = 0
- V(0) = slow eigenvector
- $V(t) = V_0\, e^{-0.2117t}$



Fast Eigenvector

- V0 = 0
- V(0) = fast eigenvector
- $V(t) = V_0\, e^{-19.65t}$



Voltages plotted every 0.05 seconds when the initial condition is the fast eigenvector

Random Initial Condition

- All eigenvectors will be excited
- The fast ones quickly decay,
- Leaving the slow eigenvector