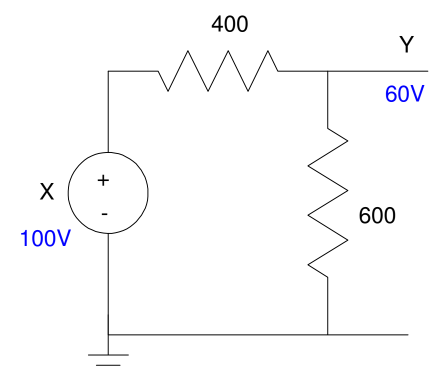
Phasors and Complex Numbers

Introduction

If you have a DC signal feeding a circuit, a single number can be used to describe the circuit. For example, consider the following circuit:



Voltage Divider: Y = 0.6X

From before, the output by voltage division is

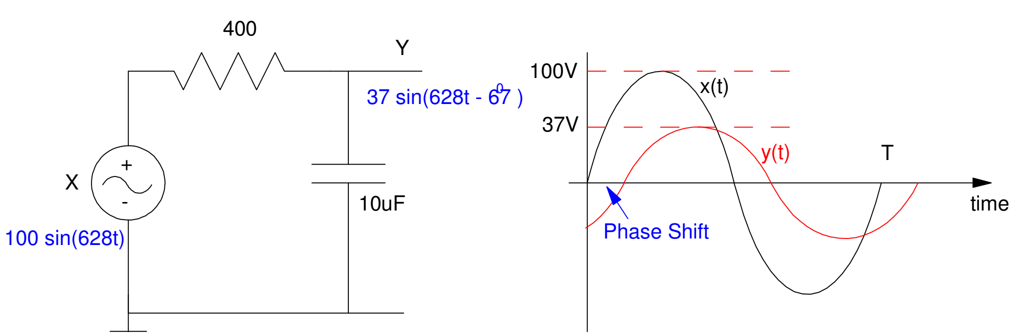
$$Y = \left(\frac{R_1}{R_1+R_2}\right)X$$

$$Y = \left(\frac{600}{600+400}\right)X$$

$$Y = 0.6 \cdot X$$

A good way to describe this circuit is to say it has a gain of 0.6: the output is 0.6 of the input.

If you change the circuit to be a capacitor and X(t) to be a sinusoid, however, the input / output relationship becomes a little more complicated



If the input is a 628 rad/sec sine wave, it is also a 100Hz sine wave

$$\omega = 2\pi f$$

$$f = \frac{\omega}{2\pi} = \frac{628}{2\pi} = 100Hz$$

The period is 10ms

$$T = \frac{1}{f} = \frac{1}{100Hz} = 0.01 \text{ sec}$$

If the input is sine wave, starting at t=0, with an amplitude of 100V, then

$$x(t) = 100 \sin(628t)$$

The output, however, is different than the input by both amplitude as well as a phase shift.

$$y(t) = 37 \sin(628t - 67^0)$$

You likewise need two numbers to describe this circuit

- An amplitude which tells you how much the signal is amplified, and
- An angle which tells you the phase shift of this circuit.

Complex numbers are a way to determine both of these with a single number.

Complex Numbers

Let

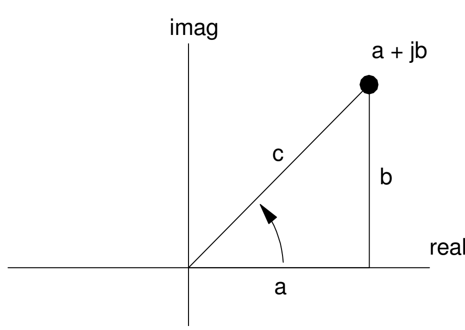
$$j = \sqrt{-1}$$

Any given number can then have a real and a complex part

$$x = a + jb$$

You can express this number in rectangular form (a + jb) or polar form

$$x = c\angle\theta$$



Complex Number (a + jb) can also be expressed as ($c\angle\theta$)

When you add complex numbers, the rectangular form is more convenient: the real part adds and the complex part adds.

$$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2$$

When you multiply complex numbers, polar form is more convenient: the amplitudes multiply, the angles add:

$$(c_1 \angle \theta_1)(c_2 \angle \theta_2) = c_1 c_2 \angle (\theta_1 + \theta_2$$

Complex numbers let you express sine and cosine with a single number as well. The complex exponential is

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

If you multiply by a complex number

$$(a + jb) \cdot e^{j\omega t} = (a + jb) \cdot (\cos(\omega t) + j \sin(\omega t))$$

$$= (a \cos(\omega t) - b \sin(\omega t)) + j(-)$$

Taking the real part you get

$$a + jb \Rightarrow a \cos(\omega t) - b \sin(\omega t)$$

The complex number a + jb represents a sine wave:

- **The real part is the cosine term**
- **The complex part is the -sine term**

With complex exponential, you can represent both sine and cosine with a single (complex) number.

Complex Impedance's

The impedance is the relationship between current and voltage. For a resistor

$$V = IR$$

The impedance of a resistor is R.

The impedance of a capacitor and inductor is a little trickier. First, assume all functions are of the form

$$x(t) = a \cdot e^{j\omega t}$$

where 'a' could be complex. The VI characteristics for a capacitor is

$$I = C \frac{dV}{dt}$$

If

$$V(t) = v_0 e^{j\omega t}$$

then

$$\frac{dV}{dt} = j\omega \cdot v_0 e^{j\omega t}$$

$$\frac{dV}{dt} = j\omega \cdot V$$

Then

$$I = C \cdot j\omega \cdot V$$

$$V = \left(\frac{1}{j\omega C}\right) I$$

The complex impedance of a capacitor is $\left(\frac{1}{j\omega C}\right)$

The impedance of an inductor is from

$$V = L\frac{dI}{dt}$$

Assuming current is of the form

$$I = i_0 \cdot e^{j\omega t}$$

then

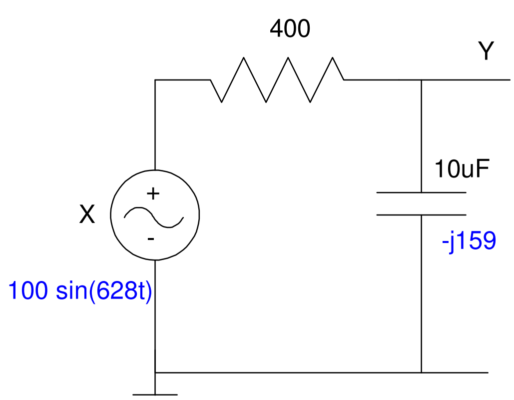
$$V = L \cdot j\omega \cdot I$$

The complex impedance of an inductor is $j\omega L$

This lets you analyze any RLC circuit the same way you analyzed a resistor circuit - only the impedance's will be complex numbers.

Example 1: RC Circuit

Determine y(t) for the following circuit:



Step 1: Replace the capacitor with its complex impedance. Since the input is 628 rad/sec, that's the frequency you care about

$$\omega = 628 \quad \text{rad/sec}$$

$$Z_c = \frac{1}{j\omega C} = -j159\Omega$$

Step 2: Solve just like you did with a DC circuit, only with complex numbers

$$Y = \left(\frac{-j159}{-j159+400}\right)X$$

$$Y = (0.37\angle-68^0)\cdot 100\sin(628t)$$

$$y(t) = 37\sin(100t-68^0)$$

In MATLAB:

```
>> C = 10e-6;
>> w = 628;
>> Zc = 1 / (j*w*C)

        0 -159.24i
>> Gain = Zc / (400 + Zc)

   0.1368 - 0.3436i


>> X = 0 - j*100;     //  x(t) = 100 sin(628t)

>> Y = Gain * X

 -34.3632 -13.6796i
```

meaning

$$y(t) = -34.3632\cos(628t) + 13.6796\sin(628t)$$

which is the rectangular form of y(t).

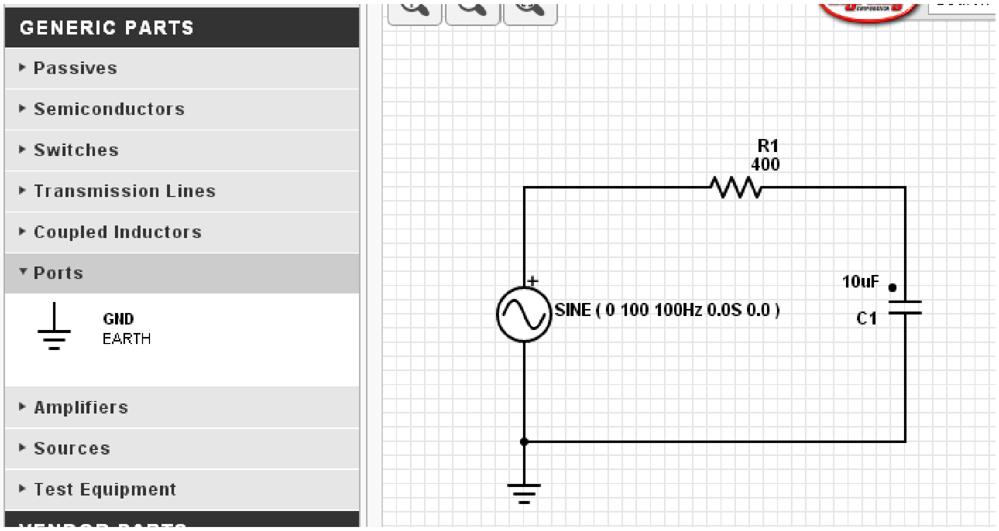
If you prefer polar form:

```
>> abs(Y)

   36.9860

>> angle(Y)*180/pi

 -158.2930
```

$$y(t) = 36.986\cos(628t-158^0)$$

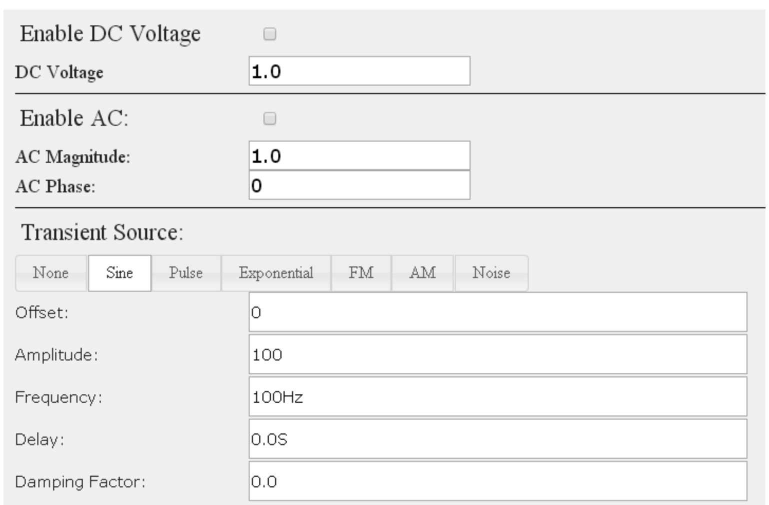
In PartSim (www.PartSim.com)

Input the circuit using drag and drop



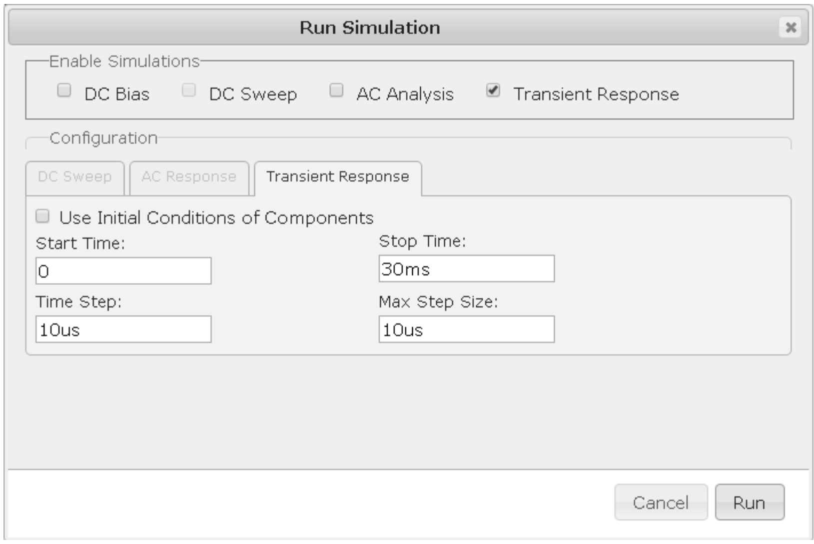
Make the input a sine wave with

- no DC offset
- 100V amplitude
- 100Hz

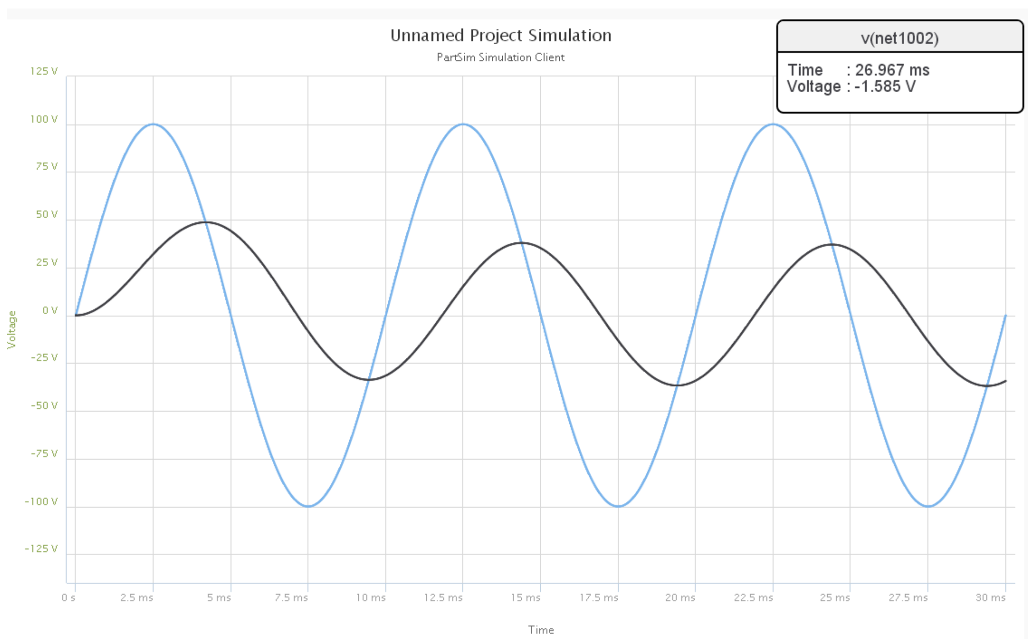


Run a transient response for

- 30ms (3 cycles)
- 10us step size



This results in a simulated input and output waveform:



Input (blue) and Output (black)

Note from the PartSim plot, the output after a short transient is

- 39Vp (as calculated)
- Delayed by 68 degrees from the input (as calculated)

To calculate the delay, use the following procedure:

- One cycle is 360 degrees or four divisions
- The output has zero crossings delayed by 0.8 divisions.

The phase shift is thus

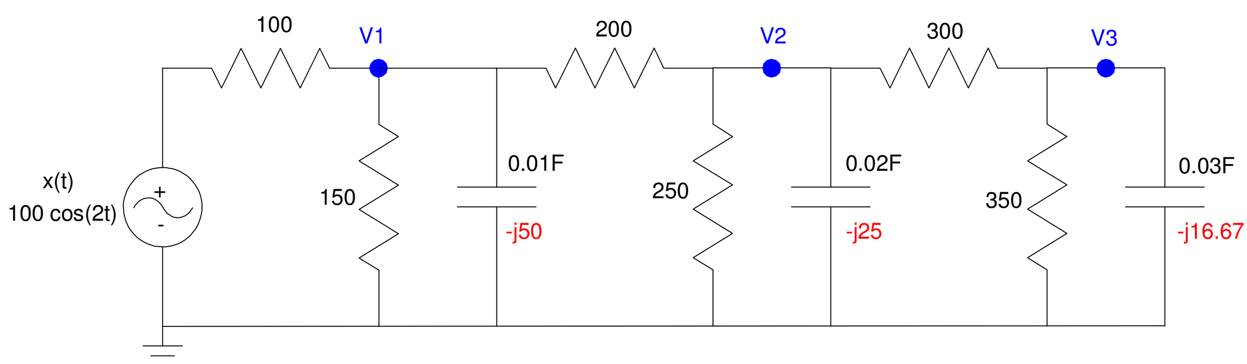
$$\left(\frac{0.8\text{div}}{4\text{ div}}\right)360^0 = 72^0 \quad \text{(actually 68 degrees but it's hard to read a graph that accurately)}$$

Negative phase shift is a delay.

Example 2: 3-Stage RC Circuit

Find the voltages for the following circuit when the input is

x(t) = 100 cos(2t)



Step 1: Change the capacitors to their complex impedance (shown in red)

$$\omega = 2$$

0.01F: $Z_c = \frac{1}{j\omega C} = -j50\Omega$

0.02F: $Z_c = \frac{1}{j\omega C} = -j25\Omega$

0.03F: $Z_c = \frac{1}{j\omega C} = -j16.67\Omega$

Step 2: Write N equations for N unknowns

V1: $\left(\frac{V_1 - X}{100}\right) + \left(\frac{V_1}{150}\right) + \left(\frac{V_1}{-j50}\right) + \left(\frac{V_1 - V_2}{200}\right) = 0$

V2: $\left(\frac{V_2 - V_1}{200}\right) + \left(\frac{V_2}{250}\right) + \left(\frac{V_2}{-j25}\right) + \left(\frac{V_2 - V_3}{300}\right) = 0$

V3: $\left(\frac{V_3 - V_2}{300}\right) + \left(\frac{V_3}{350}\right) + \left(\frac{V_3}{-j16.67}\right) = 0$

Step 3: Solve.

First, group terms

$$\left(\frac{1}{100}+\frac{1}{150}+\frac{1}{-j50}+\frac{1}{200}\right)V_1+\left(\frac{-1}{200}\right)V_2=\left(\frac{1}{100}\right)X$$

$$\left(\frac{-1}{200}\right)V_1+\left(\frac{1}{200}+\frac{1}{250}+\frac{1}{-j25}+\frac{1}{300}\right)V_2+\left(\frac{-1}{300}\right)V_3=0$$

$$\left(\frac{-1}{300}\right)V_2+\left(\frac{1}{300}+\frac{1}{350}+\frac{1}{-j16.67}\right)V_3=0$$

Place in matrix form

$$\begin{bmatrix} \left(\frac{1}{100}+\frac{1}{150}+\frac{1}{-j50}+\frac{1}{200}\right) & \left(\frac{-1}{200}\right) & 0 \\ \left(\frac{-1}{200}\right) & \left(\frac{1}{200}+\frac{1}{250}+\frac{1}{-j25}+\frac{1}{300}\right) & \left(\frac{-1}{300}\right) \\ 0 & \left(\frac{-1}{300}\right) & \left(\frac{1}{300}+\frac{1}{350}+\frac{1}{-j16.67}\right) \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\begin{bmatrix} \left(\frac{1}{100}\right) \\ 0 \\ 0 \end{bmatrix}X$$

Put into MATLAB and solve

```
>> a11 = 1/100 + 1/150 - 1/(j*50) + 1/200;
>> a12 = -1/200;
>> a13 = 0;
>> a21 = -1/200;
>> a22 = 1/200 + 1/250 -1/(j*25) + 1/300;
>> a23 = -1/300;
>> a31 = 0;
>> a32 = -1/300;
>> a33 = 1/300+1/350-1/(j*16.67);
>> A = [a11,a12,a13;a21,a22,a23;a31,a32,a33]

   0.0217 + 0.0200i  -0.0050                   0
  -0.0050             0.0123 + 0.0400i  -0.0033
        0            -0.0033              0.0062 + 0.0600i


>> B = [1/100;0;0]

     0.0100
          0
          0

>> V = inv(A)*B*X

  24.2853 -23.2416i
  -1.7971 - 3.5726i
  -0.2066 + 0.0785i
```

meaning

$$V_1(t) = 24.28\cos(2t) + 23.24\sin(2t)$$

$$V_2(t) = -1.79\cos(2t) + 3.57\sin(2t)$$

$$V_1(t) = 0.21\cos(2t) + 0.08\sin(2t)$$

If you prefer polar representation:

```
>>abs(V)

   33.6147
    3.9991
    0.2210

>> angle(V)*180/pi

  -43.7420
 -116.7040
  159.1878
```

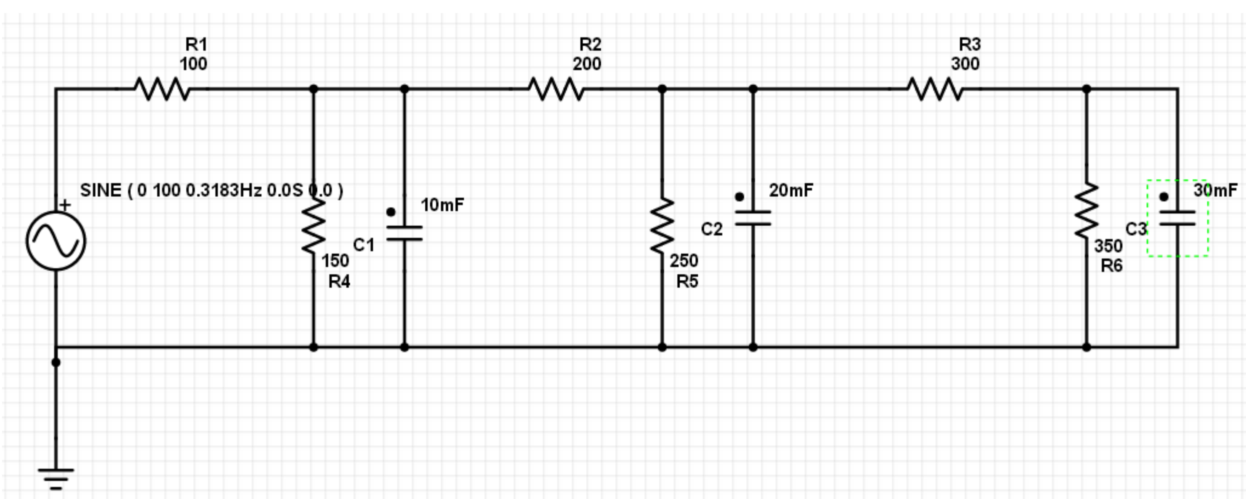
meaning

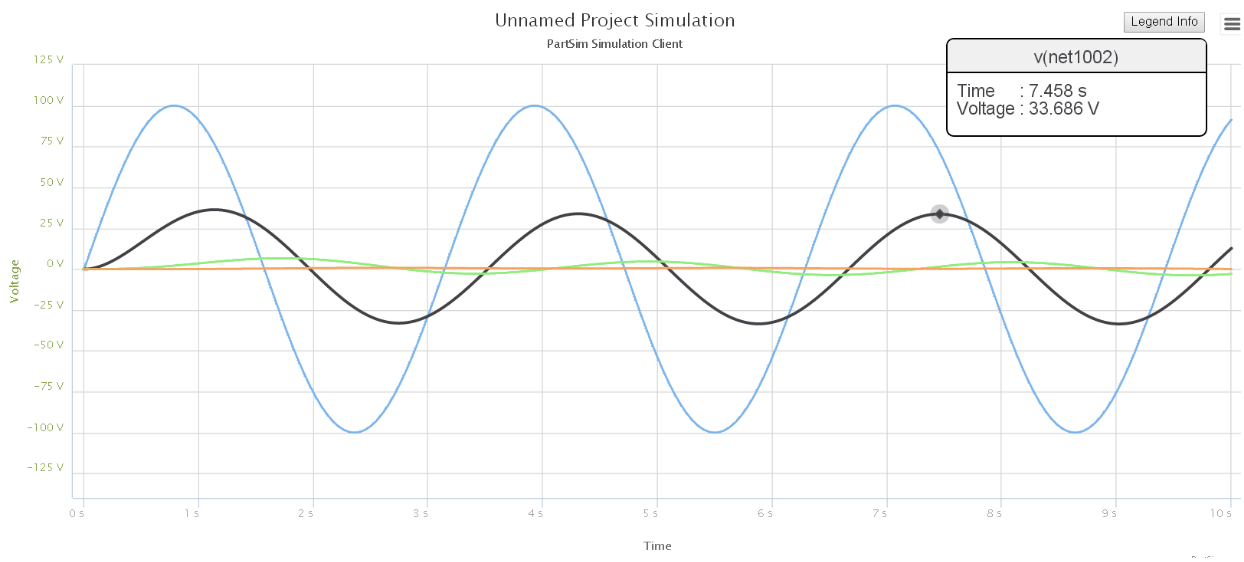
$$V_1(t) = 33.61\cos(2t - 43.7^0)$$

$$V_2(t) = 3.999\cos(2t - 116.7^0)$$

$$V_3(t) = 0.22\cos(2t + 159.2^0)$$

PartSim Simulation





Transient Response of Vin (blue), V1 (black), V2 (green), and V3 (orange)

Note that V1 has

- A peak of 33.686V (vs. 33.61V computed)
- A delay of

$$\left(\frac{0.4\text{ div}}{3.2\text{ div}}\right)360^0 = 45^0 \quad \text{(vs. 43.7 degrees computed)}$$

Likewise, V2 and V3 match our calculations

	Vin	V1	V2	V3
Calculated	100V	33.61V	3.999V	220mV
PartSim	100V	33.678V	4.233V	592 mV