Differential Equations and Transfer Functions

Objective:

- Be able to find the transfer function for a system guven its differential equation
- Be able to find the differential equation which describes a system given its transfer function.

Converting from a Differential Eqution to a Transfer Function:

Suppose you have a linear differential equation of the form:

(1) $$a_3 \frac{d^3y}{dt^3} + a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_3 \frac{d^3x}{dt^3} + b_2 \frac{d^2x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x$$

Find the forced response.

Assume all functions are in the form of est. If so, then

$$y = \alpha \cdot e^{st}$$

If you differentiate y:

$$\frac{dy}{dt} = s \cdot \alpha e^{st} = sy$$

If you differentiate again:

$$\frac{d^2y}{dt^2} = s^2 \cdot \alpha e^{st} = s^2 y$$

With the assumption that all functions are in the form of est, you can replace diffetentiation with multiplication by 's'. The s-operator then means 'the derivative of'.

Rewriting (1) using the s-operator:

$$a_3 s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 X + b_2 s^2 X + b_1 s X + b_0 X$$

(note: I've switched to capital letters to denote I'm in the s-domain. More on this when we get to LaPlace transforms.)

Solving for Y:

$$Y = \left(\frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right) X$$

or

$$Y = G(s) \cdot X$$

G(s) called the transfer function of the system and defines the gain from X to Y for all 's'.

To convert form a diffetential equation to a transfer function, replace each derivative with 's'. Rewrite in the form of Y = G(s)X. G(s) is the transfer function.

To convert to phasor notation replace

$$s \to j\omega$$

since for sinusoidal inputs, all functions are in the form of $e^{j\omega t}$. I prefer using 's' instead so I don't have to keep track of all the 'j' terms. Just remember that $s = j\omega$ when you want to do phasor analysis.

Example: Find the forced response of the following differential equation

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 7y = 3\frac{d^2x}{dt^2} + 12x$$

for

$$x(t) = 2 + 3\sin(5t) + 7\cos(10t)$$

Solution: Use superposition and treat this as three separate problems:

Step 1: Find the transfer function.

Replace all derivatives with 's':

$$s^3Y + 3s^2Y + 5sY + 7Y = 3s^2X + 12X$$

Solve for Y:

$$Y = \left(\frac{3s^2+12}{s^3+3s^2+5s+7}\right)X$$

s = j0:

$$Y_1 = \left(\frac{3s^2+12}{s^3+3s^2+5s+7}\right)_{s=j0} \cdot (2)$$

$$y_1(t) = 3.43$$

s = j5:

$$Y_2 = \left(\frac{3s^2+12}{s^3+3s^2+5s+7}\right)_{s=j5} \cdot (-j3)$$

$$Y_2 = 1.51\angle -40.91^0$$

$$y_2(t) = 1.51\cos(5t - 40.91^0)$$

s = j10

$$Y_3 = \left(\frac{3s^2+12}{s^3+3s^2+5s+7}\right)_{s=j10} \cdot (7)$$

$$Y_3 = 2.03\angle -72.86^0$$

$$y_3(t) = 2.03\cos(10t - 72.86^0)$$

The total answer is then:

$$\boxed{y(t) = 3.43 + 1.51\cos(5t - 40.91^0) + 2.03\cos(10t - 72.86^0)}$$

Note that the use of capital letters while in the phaser domain is a good reminder that the complex number isn't the answer: it's just short hand notation for the amplitude and phase shift at that frequency. The answer should be in terms of sine and cosine with real coefficients.

Going from transfer functions to differential equtions:

Given G(s), the corresponding differntial equation is easy to find. For example,

$$Y = \left(\frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}\right) X$$

First, cross multiply

$$(a_3 s^3 + a_2 s^2 + a_1 s + a_0) Y = (b_3 s^3 + b_2 s^2 + b_1 s + b_0) X$$

Multiply through:

$$a_3 s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 X + b_2 s^2 X + b_1 s X + b_0 X$$

Replace each 's' with $\frac{d}{dt}$

$$a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_3 \frac{d^3 x}{dt^3} + b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x$$

Once you get use to it, you can write down the differential equation almost by inspection. The denominator is the differential equation related to Y, the numberator is the differential equation related to X. The power on 's' is the derivative number.

Example: A system has the following transfer function:

$$Y = \left(\frac{3s+2}{s^2+7s+20}\right) X$$

What is the system's differential equation:

By inspection:

$$\boxed{\frac{d^2 y}{dt^2} + 7\frac{dy}{dt} + 20y = 3\frac{dx}{dt} + 2x}$$