Objective:

- Be able to find the transfer function for a system given its differential equation
- Be able to find the differential equation which describes a system given its transfer function.

Converting from a Differential Equation to a Transfer Function:

Suppose you have a linear differential equation of the form:

\[ a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_3 \frac{d^3 x}{dt^3} + b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x \]

Find the forced response.

Assume all functions are in the form of \( e^{st} \). If so, then

\[ y = \alpha \cdot e^{st} \]

If you differentiate \( y \):

\[ \frac{dy}{dt} = s \cdot \alpha e^{st} = sy \]

If you differentiate again:

\[ \frac{d^2 y}{dt^2} = s^2 \cdot \alpha e^{st} = s^2 y \]

With the assumption that all functions are in the form of \( e^{st} \), you can replace differentiation with multiplication by 's'. The s-operator then means 'the derivative of'.

Rewriting (1) using the s-operator:

\[ a_3 s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 X + b_2 s^2 X + b_1 s X + b_0 X \]

(note: I've switched to capital letters to denote I'm in the s-domain. More on this when we get to LaPlace transforms.)

Solving for \( Y \):

\[ Y = \left( \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right) X \]

or

\[ Y = G(s) \cdot X \]

\( G(s) \) called the transfer function of the system and defines the gain from \( X \) to \( Y \) for all 's'.

To convert from a differential equation to a transfer function, replace each derivative with 's'. Rewrite in the form of \( Y = G(s)X \). \( G(s) \) is the transfer function.

To convert to phasor notation replace
$s \to j\omega$

since for sinusoidal inputs, all functions are in the form of $e^{j\omega t}$. I prefer using 's' instead so I don't have to keep track of all the 'j' terms. Just remember that $s = j\omega$ when you want to do phasor analysis.

Example: Find the forced response of the following differential equation

$$\frac{d^3y}{dt^3} + 3 \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 7y = 3 \frac{dx}{dt^2} + 12x$$

for

$$x(t) = 2 + 3 \sin(5t) + 7 \cos(10t)$$

Solution: Use superposition and treat this as three separate problems:

Step 1: Find the transfer function.

Replace all derivatives with 's':

$$s^3 Y + 3 s^2 Y + 5 s Y + 7 Y = 3 s^2 X + 12 X$$

Solve for $Y$:

$$Y = \left( \frac{3s^2 + 12}{s^3 + 3s^2 + 5s + 7} \right) X$$

$s = j0$:

$$Y_1 = \left( \frac{3s^2 + 12}{s^3 + 3s^2 + 5s + 7} \right)_{s=0} \cdot (2)$$

$$y_1(t) = 3.43$$

$s = j5$:

$$Y_2 = \left( \frac{3s^2 + 12}{s^3 + 3s^2 + 5s + 7} \right)_{s=j5} \cdot (-j/3)$$

$$Y_2 = 1.51 \angle -40.91^0$$

$$y_2(t) = 1.51 \cos(5t - 40.91^0)$$

$s = j10$

$$Y_3 = \left( \frac{3s^2 + 12}{s^3 + 3s^2 + 5s + 7} \right)_{s=j10} \cdot (7)$$

$$Y_3 = 2.03 \angle -72.86^0$$

$$y_3(t) = 2.03 \cos(10t - 72.86^0)$$

The total answer is then:
\[ y(t) = 3.43 + 1.51 \cos(5t - 40.91^0) + 2.03 \cos(10t - 72.86^0) \]

Note that the use of capital letters while in the phaser domain is a good reminder that the complex number isn't the answer: it's just shorthand notation for the amplitude and phase shift at that frequency. The answer should be in terms of sine and cosine with real coefficients.

**Going from transfer functions to differential equations:**

Given \( G(s) \), the corresponding differential equation is easy to find. For example,

\[ Y = \left( \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \right) X \]

First, cross multiply

\[ (a_3 s^3 + a_2 s^2 + a_1 s + a_0) Y = (b_3 s^3 + b_2 s^2 + b_1 s + b_0) X \]

Multiply through:

\[ a_3 s^3 Y + a_2 s^2 Y + a_1 s Y + a_0 Y = b_3 s^3 X + b_2 s^2 X + b_1 s X + b_0 X \]

Replace each 's' with \( \frac{d}{dt} \)

\[ a_3 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_3 \frac{d^3 x}{dt^3} + b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x \]

Once you get used to it, you can write down the differential equation almost by inspection. The denominator is the differential equation related to \( Y \), the numerator is the differential equation related to \( X \). The power on 's' is the derivative number.

Example: A system has the following transfer function:

\[ Y = \left( \frac{3s^2}{s^2 + 7s + 20} \right) X \]

What is the system's differential equation:

By inspection:

\[ \frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 20y = 3 \frac{dx}{dt} + 2x \]