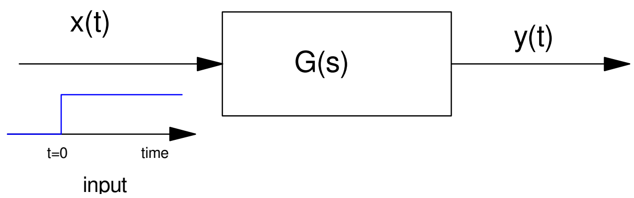
Transfer Functions and Forced Response

Background

Given a dynamic system (i.e. a system which is described by a differential equation),



the method you use to find the output depends upon the input:

- If the input is a sinusoid, use phasor analysis.
- If the input is periodic in time T, use Fourier transforms.
- If the input is non-periodic and causal (zero for t<0), use LaPlace transforms.

Transfer Functions

Assume you have a dynamic system

$$y''' + a_2 y'' + a_1 y' + a_0 y = b_2 x'' + b_1 x' + b_0 x$$

Also assume that x(t) is zero for t<0. In this case, the initial conditions will all be zero, making the LaPlace transform

$$(s^3 + a_2 s^2 + a_1 s + a_0)Y = (b_2 s^2 + b_1 s + b_0)X$$

Solving for Y

$$Y = \left(\frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \right) X$$

or

$$Y = G(s)X$$

where G(s) is called *the Transfer Function from X to Y*

Note that the fundamental assumption behind LaPlace transforms is that all functions are in the form of

$$y(t) = a \cdot e^{st}$$

When you differentiate, you get

$$\frac{dy}{dt} = s \cdot ae^{st} = sY$$

so the notation 'sY' can be read as 'the derivative of y'. A 3rd-order transfer function in 's' means you are looking at a 3rd-order differential equation.

Step Response

Given a dynamic system

$$Y = G(s) \cdot X$$

if x(t) is causal (zero for t<0), then you can solve for y(t) by

- Finding X(s), the LaPlace transform for x(t),
- Multiply G(s) by X(s) to find Y(s), then
- Taking the inverse-LaPlace transform to find y(t).

Example 1: Find y(t) assuming x and y are related by

$$Y = \left(\frac{3}{s+2}\right)X$$

and

$$x(t) = u(t)$$

Solution: Take the LaPlace transform of x(t)

$$X(s) = \left(\frac{1}{s}\right)$$

Find Y(s)

$$Y = \left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)$$

Take the inverse-LaPlace transform

$$Y = \left(\frac{3}{s(s+2)}\right) = \left(\frac{a}{s}\right) + \left(\frac{b}{s+2}\right)$$

$$a = \left(\frac{3}{(s+2)}\right)_{s \to 0} = 1.5$$

$$b = \left(\frac{3}{s}\right)_{s \to -2} = -1.5$$

so

$$Y = \left(\frac{1.5}{s}\right) + \left(\frac{-1.5}{s+2}\right)$$

and

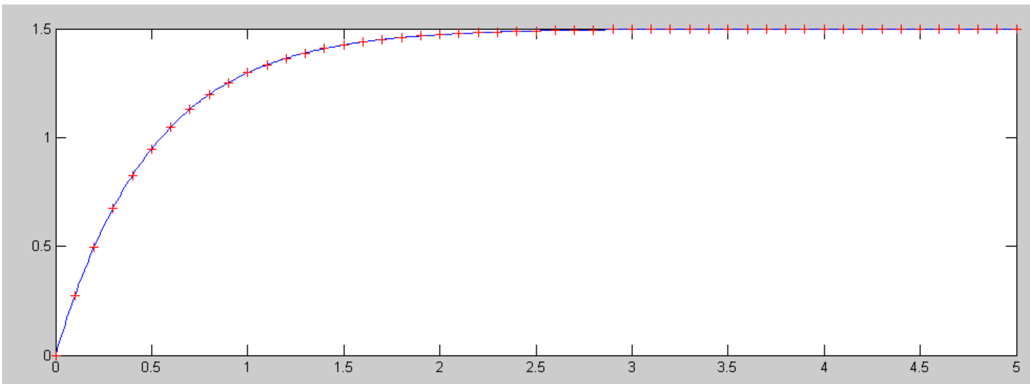
$$y(t) = (1.5 - 1.5e^{-2t})u(t)$$

Checking your answer in Matlab:

```
t = [0:0.01:5]';
G = tf(3,[1,2])


   3
-----
s + 2

y = step(G,t);
y1 = 1.5 - 1.5*exp(-2*t);
N = [1:10:length(t)]';
plot(t,y,'-',t(N),y1(N),'r+')
```



Step Response Computed in Matlab (blue) and by hand (red)

Note that the first term is also the phasor solution for a DC input

$$x(t) = 1$$

$$Y(j\omega) = G(j\omega) \cdot X(j\omega)$$

$$Y(j0) = G(j0) \cdot X(j0)$$

$$Y = \left(\frac{3}{s+2}\right)_{s=0} \cdot 1$$

$$Y = 1.5$$

Phasors tell you the steady-state solution (they assume the input has been on for all time).

LaPlace transforms tell you

- The steady-state solution (as time goes to infinity), and
- The transient solution (how you go from zero at t=0 to the steady-state solution).

So,

- Phasors are actually a special case of LaPlace transforms, and
- You *can* use LaPlace transforms to find the steady-state solution, but it's a lot harder.

Example 2: Find y(t) given

$$Y = \left(\frac{2s+100}{s^3+7s^2+20s+50}\right)X$$

$$x(t) = u(t)$$

Solution: Take the LaPlace transform for x(t)

$$X = \frac{1}{s}$$

Find Y(s)

$$Y = \left(\frac{2s+100}{s^3+7s^2+20s+50}\right)\left(\frac{1}{s}\right)$$

find y(t)

$$Y(s) = \left(\frac{2s+100}{s(s+1+j3)(s+1-j3)(s+5)}\right)$$

$$Y(s) = \left(\frac{2s+100}{s(s+1+j3)(s+1-j3)(s+5)}\right) = \left(\frac{a}{s}\right) + \left(\frac{b}{s+1+j3}\right) + \left(\frac{c}{s+1-j3}\right) + \left(\frac{d}{s+5}\right)$$

$$a = \left(\frac{2s+100}{(s+1+j3)(s+1-j3)(s+5)}\right)_{s=0} = 2$$

$$b = \left(\frac{2s+100}{s(s+1-j3)(s+5)}\right)_{s=-1-j3} = 1.0349\angle-128.2^0$$

$$c = \left(\frac{2s+100}{s(s+1-j3)(s+5)}\right)_{s=-1+j3} = 1.0349\angle128.2^0$$

$$d = \left(\frac{2s+100}{s(s+1+j3)(s+1-j3)}\right)_{s=-5} = -0.72$$

so

$$Y(s) = \left(\frac{2}{s}\right) + \left(\frac{1.0349\angle-128.2^0}{s+1+j3}\right) + \left(\frac{1.0349\angle128.2^0}{s+1-j3}\right) + \left(\frac{-0.72}{s+5}\right)$$

and

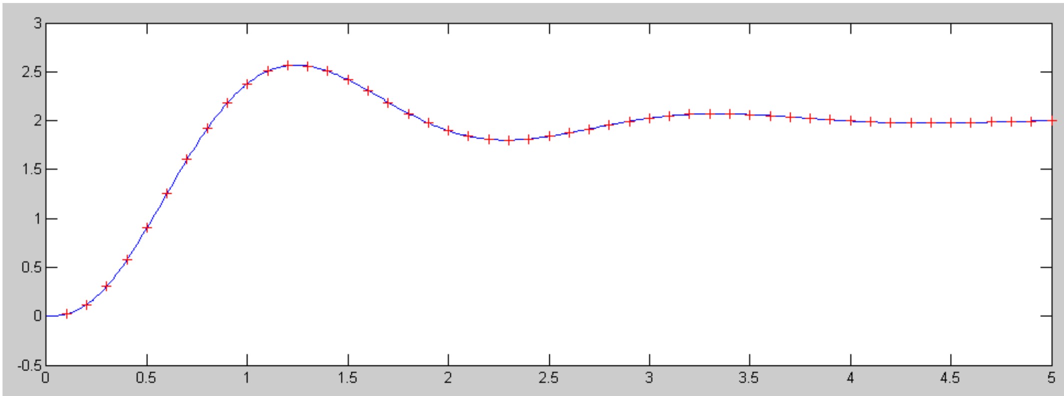
$$\boxed{y(t) = (2 + 2.0699e^{-t}\cos(3t + 128.2^0) - 0.72e^{-5t})u(t)}$$

Checking in Matlab

```
G = tf([2,100],[1,7,20,50])

        2 s + 100
-----------------------
s^3 + 7 s^2 + 20 s + 50
```

```
t = [0:0.01:5]';
y = step(G,t);
y1 = 2 + 2.0699*exp(-t) .* cos(3*t + 2.238) - 0.72*exp(-5*t);
N = [1:10:length(t)]';
plot(t,y,'-',t(N),y1(N),'r+')
```



Step Response Computed in Matlab (blue) and by hand (red)

Again, the steady-state solution is also the phasor solution. At DC

$$Y(j0) = \left(\frac{2s+100}{s^3+7s^2+20s+50}\right)_{s=0} \cdot X(j0)$$

$$Y = 2 \cdot 1$$

The steady-state solution (as time goes to infinity) is

$$y(t) = 2$$

Response for Other Inputs:

Find y(t) for

$$Y = \left(\frac{3}{s+2}\right)X$$

$$x(t) = \cos(4t)u(t)$$

Solution: Find the LaPlace transform for x(t)

$$X(s) = \left(\frac{0.5}{s+j4}\right) + \left(\frac{0.5}{s-j4}\right)$$

Putting over a common denominator

$$X(s) = \left(\frac{0.5}{s+j4}\right)\left(\frac{s-j4}{s-j4}\right) + \left(\frac{0.5}{s-j4}\right)\left(\frac{s+j4}{s+j4}\right)$$

$$X(s) = \left(\frac{s}{s^2+16}\right)$$

Find Y(s)

$$Y = \left(\frac{3}{s+2}\right)X$$

$$Y = \left(\frac{3}{s+2}\right)\left(\frac{s}{s^2+16}\right)$$

Find y(t)

$$Y = \left(\frac{3s}{(s+2)(s+j4)(s-j4)}\right) = \left(\frac{a}{s+2}\right) + \left(\frac{b}{s+j4}\right) + \left(\frac{c}{s-j4}\right)$$

$$a = \left(\frac{3s}{(s+j4)(s-j4)}\right)_{s=-2} = -0.300$$

$$b = \left(\frac{3s}{(s+2)(s-j4)}\right)_{s=-j4} = 0.3354\angle 63.43^0$$

$$c = \left(\frac{3s}{(s+2)(s+j4)}\right)_{s=j4} = 0.3354\angle -63.43^0$$

so

$$Y = \left(\frac{-0.300}{s+2}\right) + \left(\frac{0.3354\angle 63.43^0}{s+j4}\right) + \left(\frac{0.3354\angle -63.43^0}{s-j4}\right)$$

and

$$\boxed{y(t) = (-0.3e^{-2t} + 0.6708\cos(4t - 63.43^0))u(t)}$$

Checking in Matlab: Matlab has a step function but it doesn't have a response to a 4 rad/sec cosine input function. We can still use *impulse* however.

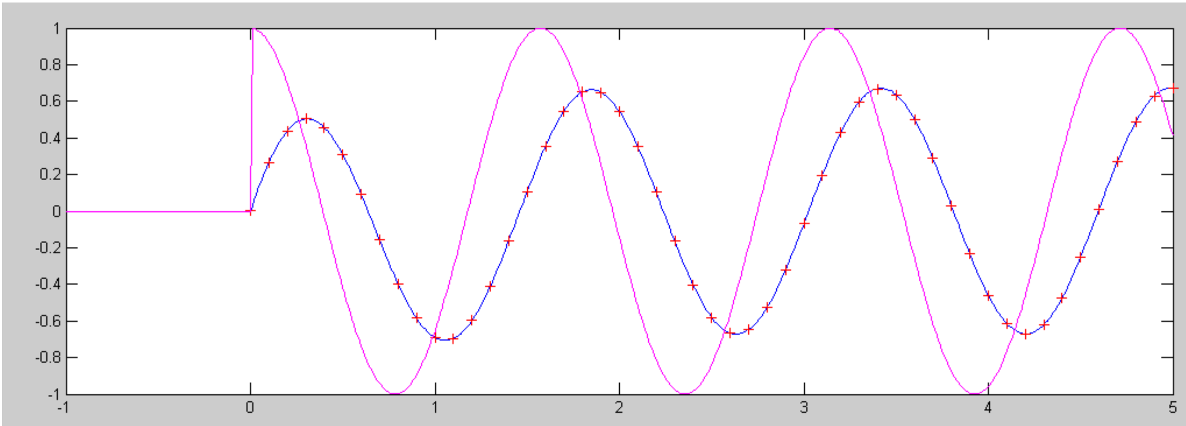
```
Y = zpk(0,[-2,j*4,-j*4],3)

       3 s
----------------
(s+2) (s^2 + 16)

t = [0:0.01:5]';
y = impulse(Y,t);

y1 = -0.3*exp(-2*t) + 0.6708*cos(4*t - 1.107);

t2 = [-1:0.01:5]';
x = cos(4*t2) .* (t2>0);
plot(t,y,'-',t(N),y1(N),'r+',t2,x,'m')
```



Input, x(t) (pink) and output as computed in matlab (blue) and by hand (red)

Note that since x(t) = 0 for t<0, there is jump at t=0. Also note that the steady-state solution matches what you compute using phasor analysis. At all frequencies,

$$Y = \left(\frac{3}{s+2}\right) X$$

Using phasor analysis at 4 rad/sec

$$Y = \left(\frac{3}{s+2}\right)_{s=j4} \cdot (1 + j0)$$

$$Y = 0.6708 \angle -63.4^0$$

which is phasor form for

$$y(t) = 0.6708 \cos(4t - 63.4^0)$$

Again, phasor analysis tells you the steady-state portion of the solution of

$$y(t) = (-0.3e^{-2t} + 0.6708 \cos(4t - 63.43^0))u(t)$$