# Transfer Functions

Transfer functions are a convenient way to describe a system which is described by a differential equation, such as a circuit with capacitors and inductors. If you have a circuit with N capacitors and inductors, you need an Nth-order differential equation to describe that circuit. This comes from the VI characteristics for inductors and capacitors:

$$v = L\tfrac{di}{dt}$$

$$i = C\tfrac{dv}{dt}$$

The basic assumption behind LaPlace transforms is that all functions are of the form

$$y(t) = e^{st}$$

With this assumption, differentiation becomes multiplication by 's'

$$\tfrac{dy}{dt} = s \cdot e^{st} = s \cdot y$$

'sy' can likewise be read as *the derivative of y'*. The reason you do this is LaPlace transforms convert differential equations into algebraic equations in 's' - with the presumption that algebra is easier than calculus. The transfer function, G(s), is just the function of 's' which relates the input and output:

$$Y = G(s) \cdot X$$

Example: Find the transfer function from X to Y for the following dynamic systems (i.e. systems which are described by differential equations)

- $\frac{dy}{dt} + 3y = 6x$
- $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 3\frac{dx}{dt} + 8x$
- $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 2\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 20y$

Solution: Replace d/dt with 's' and solve for Y in term of x

i) $$\tfrac{dy}{dt} + 3y = 6x$$

$$sY = 3Y = 6X$$

$$Y = \left(\tfrac{6}{s+3}\right)X$$

$$\boxed{G(s) = \left(\tfrac{6}{s+3}\right)}$$

ii) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 3\frac{dx}{dt} + 8x$

$$s^2Y + 5sY + 4Y = 3sX + 8X$$

$$Y = \left(\frac{3s+8}{s^2+5s+4}\right)X$$

$$\boxed{G(s) = \left(\frac{3s+8}{s^2+5s+4}\right)}$$

iii) $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 2\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 20x$

$$s^3Y + 6s^2Y + 11sY + 6Y = 2s^2X + 9sX + 20X$$

$$Y = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)X$$

$$\boxed{G(s) = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)}$$

Transfer functions are convenient to use and let you find the output for various inputs.

Example 2: Assume X and Y are related by

$$Y = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)X$$

Find y(t) assuming

- $x(t) = 3\sin(4t)$
- $x(t) = 3e^{2t}\sin(4t)$
- $x(t) = 3u(t)$
- $x(t) = 3\sin(4t)u(t)$

Solution: The approach you take depends upon the input.

i) $x(t) = 3\sin(4t)$

This is actually a phasor problem: the input has been on for all time so all we care about is the steady-state solution. Note that G(s) is true for all 's'. In this case, all we care about is

$$s = j4$$

$$x(t) = 3\sin(4t) \rightarrow 0 - j3$$

( real is cosine, -imag is sine )

$$G(j4) = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)_{s=j4} = 0.4116\angle -84^0$$

$$Y = G \cdot X = (0.4116\angle -84^0) \cdot (0-j3)$$

$$Y = 1.234\angle -174^0$$

This is phasor shorthand for

$$\boxed{y(t) = 1.234\cos(4t - 174^0)}$$

ii) $x(t) = 3e^{2t}\sin(4t)$

This isn't normally done but phasors also work for this type of input (which has been on for all time). Here

$$s = 2 + j4$$

(the real part is the exponential, the complex part is the sine wave).

$$X = 0 - j3$$

$$G(j4) = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)_{s=2+j4} = 0.3833\angle -58^0$$

$$Y = G \cdot X$$

$$Y = (0.3833\angle -58^0) \cdot (0-j3)$$

$$Y = 1.150\angle -148^0$$

which is phasor shorthand for

$$\boxed{y(t) = 1.15e^{2t}\cos(4t - 148^0)}$$

iii) $x(t) = 3u(t)$

This is actually a LaPlace problem since x(t) = 0 for t<0. Taking the LaPlace transform

$$X = \frac{3}{s}$$

then

$$Y = G \cdot X$$

$$Y = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)\left(\frac{3}{s}\right)$$

Factor

$$Y = \left(\frac{6s^2+27s+60}{s(s+1)(s+2)(s+3)}\right)$$

Take the partial fraction expansion

$$Y = \left(\frac{10}{s}\right) + \left(\frac{-19.5}{s+1}\right) + \left(\frac{15}{s+2}\right) + \left(\frac{-5.5}{s+3}\right)$$

Take the inverse LaPlace transform

$$y(t) = (10 - 19.6e^{-t} + 15e^{-2t} - 5.5e^{-3t})u(t)$$

iv) $x(t) = 3\sin(4t)u(t)$

Again, this is a LaPlace problem since x(t) = 0 for t<0. Taking the LaPlace transform and using Wikipedia:

$$\sin(\omega t) \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$\cos(\omega t) \leftrightarrow \frac{s}{s^2+\omega^2}$$

gives

$$X(s) = \left(\frac{12}{s^2+16}\right)$$

and

$$Y = \left(\frac{2s^2+9s+20}{s^3+6s^2+11s+6}\right)\left(\frac{12}{s^2+16}\right)$$

Factoring

$$Y = \left(\frac{2s^2+9s+20}{(s+1)(s+2)(s+3)}\right)\left(\frac{12}{(s+j4)(s-j4)}\right)$$

$$Y = \left(\frac{4.5882}{s+1}\right) + \left(\frac{-6}{s+2}\right) + \left(\frac{2.64}{s+3}\right) + \left(\frac{0.6174\angle 174^0}{s+j4}\right) + \left(\frac{0.6174\angle -174^0}{s-j4}\right)$$

Taking the inverse LaPlace transform

$$\boxed{y(t) = (4.5882e^{-t} - 6e^{-2t} + 2.64e^{-3t} + 1.234\cos(4t - 174^0))u(t)}$$

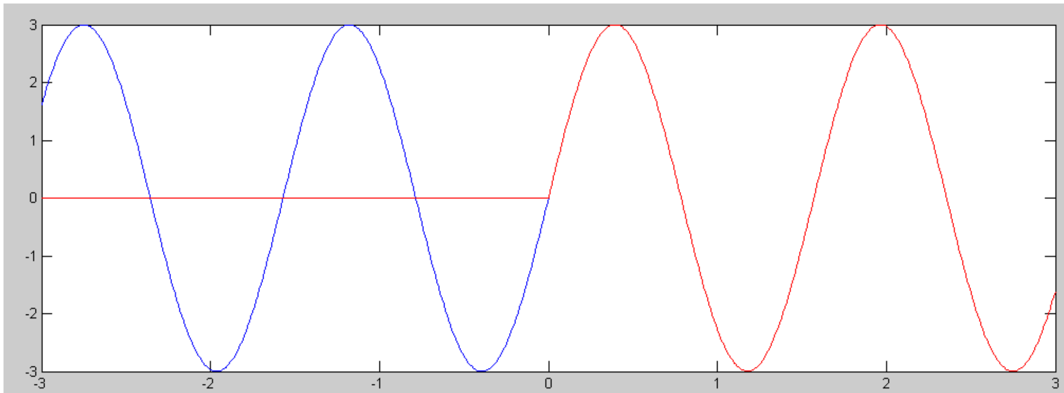
Note that the last term matches what we calculated using phasor analysis. You *can* use LaPlace transforms to find the response for a dynamic system with a sinusoidal input. It's a lot harder though.

The difference between part i) and part iv) is

- Part i) assumes x(t) is on for all time (so we use phasor analysis)
- Part iv) assumes x(t) is zero for t<0 (so we use LaPlace tranforms).

The two inputs are as follows

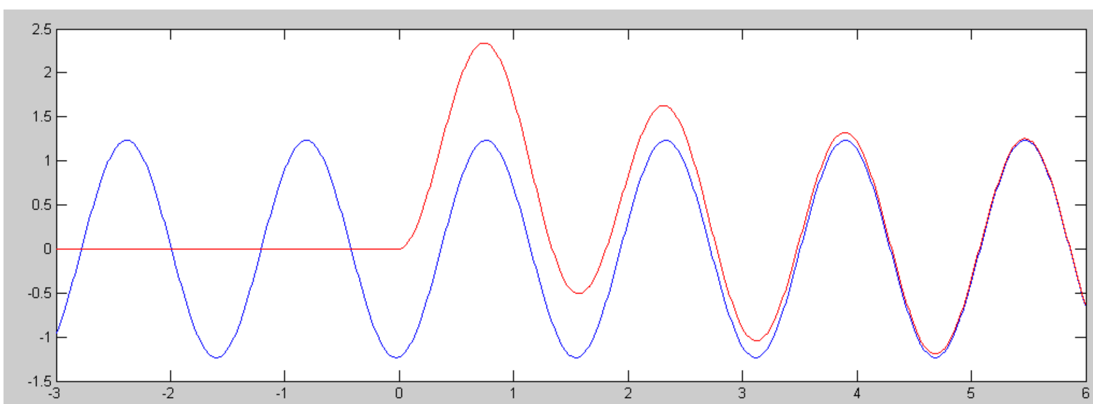
```
t = [-3:0.01:3]';
x1 = 3*sin(4*t);
x4 = 3*sin(4*t) .* (t>0);
plot(t,x1,'b',t,x4,'r');
```




x(t) for part (i) (blue) and part iv (red)

The output is

```
t = [-3:0.01:3]';
y1 = 1.234*cos(4*t - 3.03);
y4 = 4.5882*exp(-t) - 6*exp(-2*t) + 2.64*exp(-3*t) +1.234*cos(4*t - 3.03);
plot(t,y1,'b',t,y2,'r');
```




y(t) for part (i) (blue) and part iv (red)

Note for the output, the two results are the same as time goes to infinity. LaPlace transforms show you what happens during the transient (the time right after t=0) when you turn on the system.

- If you care about the transient response, you need to use LaPlace transforms.
- If you don't care about the transient response and only care about the steady-state response, use phasors.