Passive RL and RC Filters

A filter is a system whose gain changes with frequency. Essentially, all dynamic systems are filters.

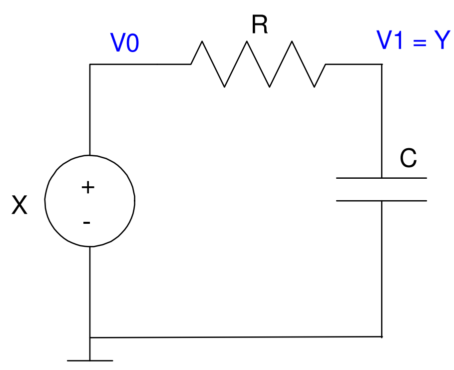
1-Stage Low-Pass Filter

For the following 1-stage RC low-pass filter with R = 100k, C = 1uF,

- Find the transfer function from X to Y
- Find y(t) assuming

$$x(t) = 5 + 2.5\sin(377t)$$

- Check your answer in PartSim



Single-Stage RC Low Pass Filter: $Y = \left(\frac{1}{RCs+1}\right)X$

i) Find the transfer function from X to Y:

The LaPlace impedance of a capacitor is

$$Z = \frac{1}{Cs}$$

By voltage division, the gain is then

$$Y = \left(\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}}\right)X = \left(\frac{1}{RCs+1}\right)X$$

This is called a *low pass filter* since

- At low frequencies ($s = j\omega = j0$), the gain is one. The input signal is passed to the output.
- At high frequencies (s large) the gain goes to zero. The input signal is blocked.

ii) Find y(t) assuming

$$x(t) = 5 + 2.5\sin(377t)$$

Since RC = 0.10.

$$Y = \left(\frac{1}{0.1s+1}\right)X$$

Using superposition, treat this as two separate problems

$$x(t) = 5$$
$$X = 5$$
$$s = j\omega = j0$$
$$Y = \left(\frac{1}{0.1s+1}\right)_{s=0} \cdot 5$$
$$Y = 5$$
$$y(t) = 5$$

$$x(t) = 2.5\sin(377t)$$
$$X = 0 - j2.5$$
$$s = j377$$
$$Y = \left(\frac{1}{0.1s+1}\right)_{s=j377} \cdot (0 - j2.5)$$
$$Y = (0.0265\angle -88^0) \cdot (-j2.5)$$
$$Y = 0.0663\angle -178^0$$
$$y(t) = 0.0663\cos(377t - 178^0)$$

The total input is the sum of the two parts. The total output is the sum of the two outputs

$$\boxed{y(t) = 5 + 0.0663\cos(377t - 178^0)}$$

The RC filter

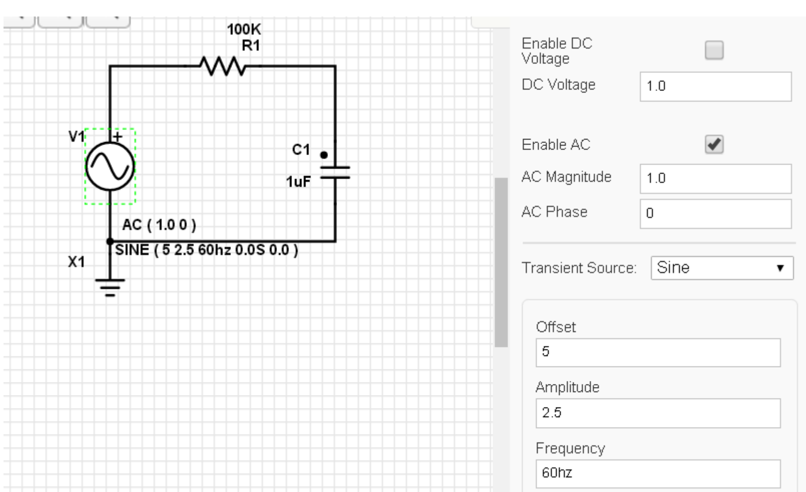
- Passed the DC term (5V), and
- Attenuated the 60Hz term from 2.5Vp to 66.3mVp

iii) Check your answer in PartSim

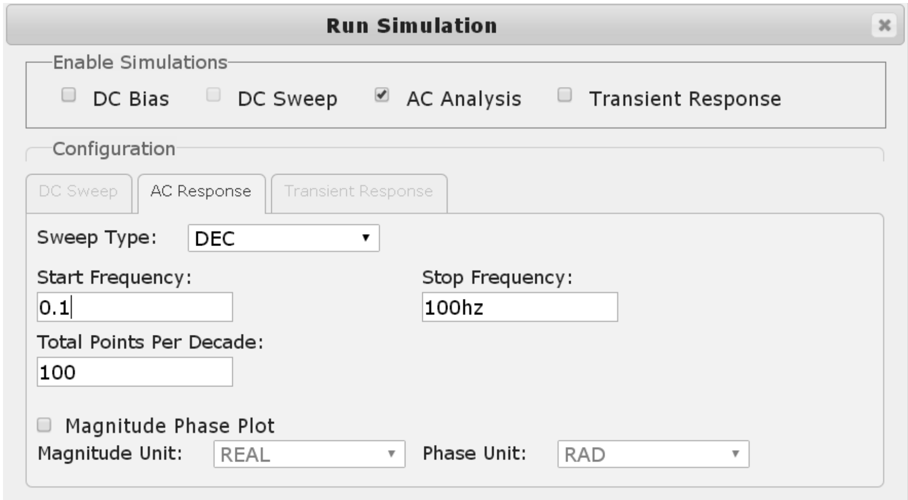
There are two ways to do this: either

- Check the gain of the filter at DC and 60Hz, or
- Check the transient response matches our calculations

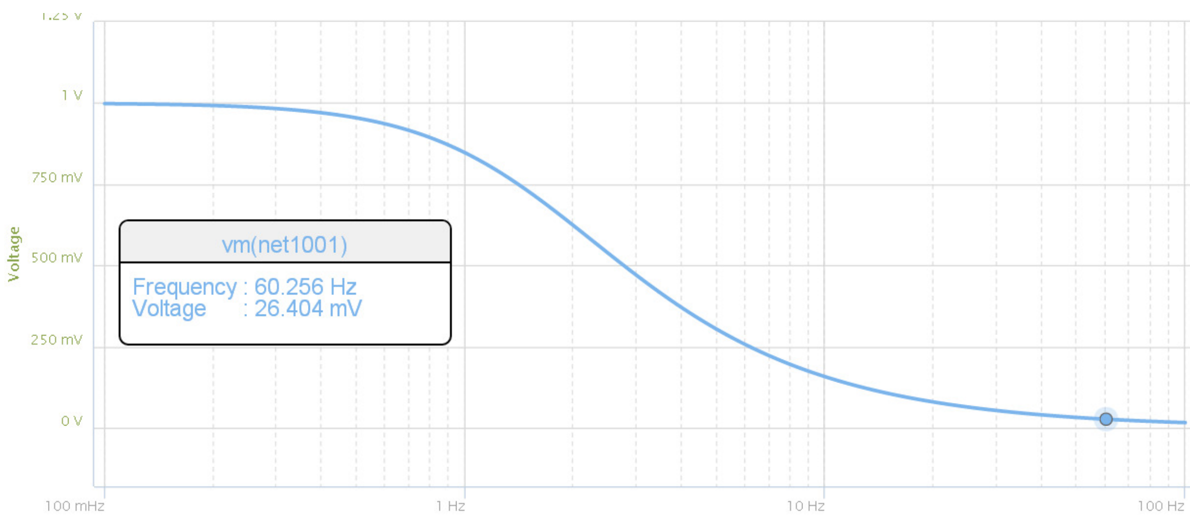
First, input the circuit into PartSim:



Run an AC Response from 0.1 to 100Hz (close to 0Hz and 60Hz)



The resulting plot looks like this:

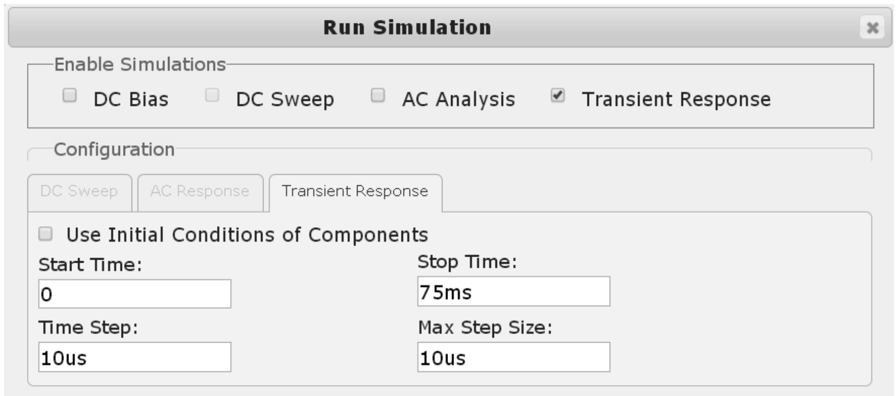


Note that

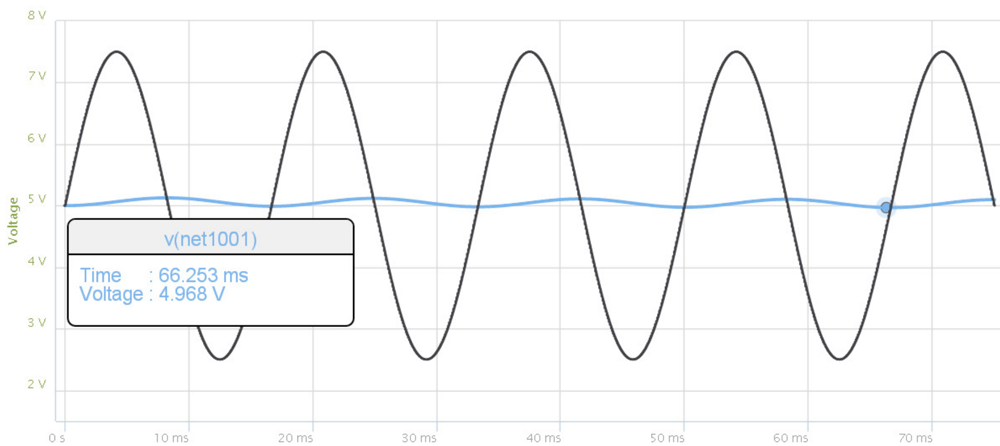
- At 0.1Hz (almost DC), the gain is 1.00 as calculated
- At 60Hz, the gain is 0.0264 (vs. 0.0265 calculated).

This matches our calculations.

A second method to check your answer is to run a transient simulation. A 60Hz signal has a period of 17ms. Running the simulation for 5 cycles (75ms) results in



Running a transient simulation to see what the output looks like:



Resulting signals at X (black) and Y (blue).

Note again that this matches our calculations

- The output has a DC offset of 5V
- With a peak-to-peak amplitude of 135mVpp (vs. 132mVpp computed)

3-Stage RC Low-Pass Filter (take 1)

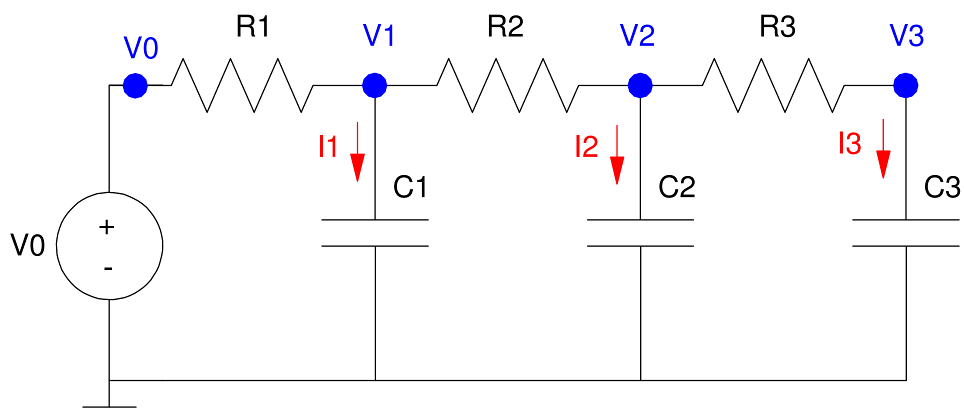
A single stage RC low-pass filter helps to remove high-frequency terms, such as the ripple in the above example. On the basis that mode is better, a 3-stage RC low pass filter should be better than a one-stage.

Example 2: For the following 3-stage RC low-pass filter with R = 100k, C = 1uF,

- Find the transfer function from X to Y
- Find y(t) assuming

$$x(t) = 5 + 2.5\sin(377t)$$

- Check your answer in PartSim



3-Stage RC Low Pass Filter

i) Find the transfer function from X to Y

Due to loading, you can't treat this as three cascaded filters. Instead, use state-space and Matlab to get the transfer function. Assuming zero initial conditions (or that initial conditions don't matter), the voltage node equations are:

$$I_1 = C_1 s V_1 = \left(\frac{V_0 - V_1}{R_1}\right) + \left(\frac{V_2 - V_1}{R_2}\right)$$

$$I_2 = C_2 s V_2 = \left(\frac{V_1 - V_2}{R_2}\right) + \left(\frac{V_3 - V_2}{R_3}\right)$$

$$I_3 = C_3 s V_3 = \left(\frac{V_2 - V_3}{R_3}\right)$$

Solving for the derivatives

$$sV_1 = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_1 + \left(\frac{1}{R_2C_1}\right)V_2 + \left(\frac{1}{R_1C_1}\right)V_0$$

$$sV_2 = -\left(\frac{1}{R_2C_2} + \frac{1}{R_3C_2}\right)V_2 + \left(\frac{1}{R_2C_2}\right)V_1 + \left(\frac{1}{R_3C_2}\right)V_3$$

$$sV_3 = -\left(\frac{1}{R_3C_3}\right)V_3 + \left(\frac{1}{R_3C_2}\right)V_2$$

Placing in matrix form:

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \left(\frac{1}{R_2C_1}\right) & 0 \\ \left(\frac{1}{R_2C_2}\right) & -\left(\frac{1}{R_2C_2} + \frac{1}{R_3C_2}\right) & \left(\frac{1}{R_3C_2}\right) \\ 0 & \left(\frac{1}{R_3C_2}\right) & -\left(\frac{1}{R_3C_3}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{R_1C_1}\right) \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = V_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Plugging in R = 100k, C = 1uF

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -20 & 10 & 0 \\ 10 & -20 & 10 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} V_0$$

Using Matlab to find the transfer function to V3:

```
>> A = [-20,10,0 ; 10,-20,10 ; 0,10,-10]

   -20    10     0
    10   -20    10
     0    10   -10

>> B = [10;0;0]

    10
     0
     0
>> C = [0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                    1000
G(s) = -----------------------------
       (s+32.47) (s+15.55) (s+1.981)
```

Note that

- You could solve for the transfer function by hand by simplifying 3 equations. Matlab is lots easier.
- For a single stage RC filter, the poles are at 1/RC = -10
- For a 3-stage RC filter, the are no longer at s = -10 due to loading.

ii) Find y(t) assuming

$$x(t) = 5 + 2.5\sin(377t)$$

Like before, use superposition and the transfer function we just found:

$$x(t) = 5$$
$$s = 0$$
$$X = 5$$
$$Y = \left(\frac{1000}{(s+32.47)(s+15.55)(s+1.981)}\right)_{s=0} \cdot X$$
$$Y = (1)\cdot(5)$$
$$Y = 5$$
$$y(t) = 5$$

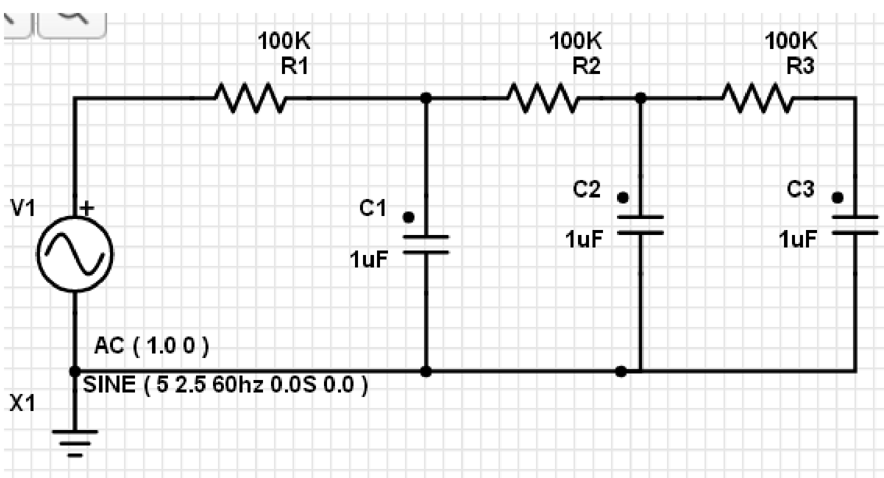
$$x(t) = 2.5\sin(377t)$$
$$s = j377$$
$$X = 0 - j2.5$$
$$Y = \left(\frac{1000}{(s+32.47)(s+15.55)(s+1.981)}\right)_{s=j377} \cdot X$$
$$Y = (0.0000185\angle 97^0)\cdot(0 - j2.5)$$
$$Y = 0.000046\angle 7^0$$
$$y(t) = 0.000046\cos(377t + 7^0)$$

Putting it all together...

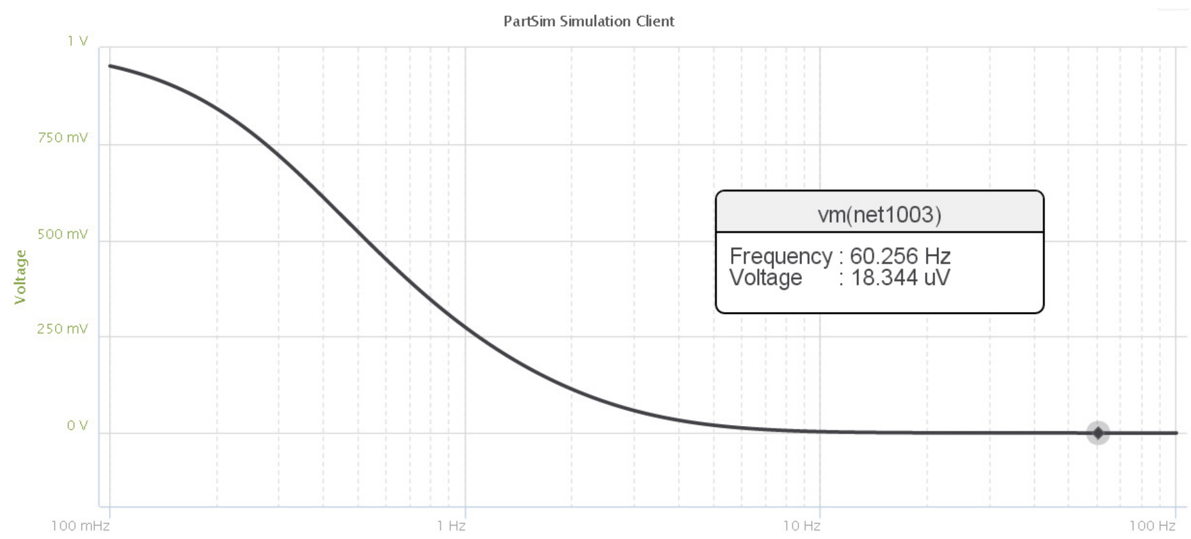
$$\boxed{y(t) = 5 + 0.000046\cos(377t + 7^0)}$$

iii) Check your answer in PartSim:

Input the circuit into PartSim:



Check the gain at DC and 60Hz:



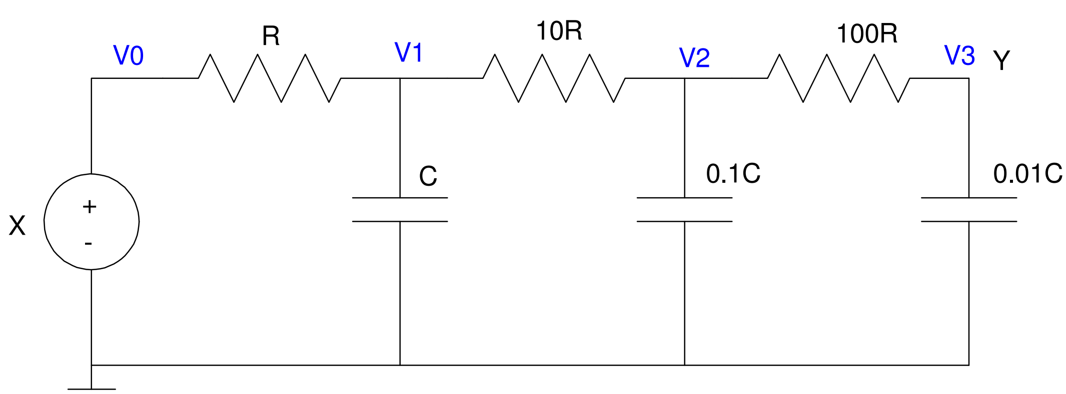
Note that

- The DC gain is almost 1 (you need to go lower than 0.1Hz to see this), and
- The gain at 60Hz is 0.000 01834 (vs. 0.000 0185 computed)

This matches our calculations.

-Stage RC Filter (take 2)

When building a 3-stage RC low-pass filter, you can't simply tack on three more stages. When analyzing the 1-stage RC filter we assumed that there was no loading. If you add an identical stage to V1, this assumption is violated and the circuit is changed. To avoid loading, one trick is to make the impedance of each stage increase by 10x relative to the previous stage:



3-Stage RC Low-Pass Filter: $Y \approx \left(\frac{1}{RCs+1}\right)^3 X$

Now, if

- R = 100k
- C = 1uF
- 1 / RC = 10

the dynamics become

$$\begin{bmatrix} sV_1 \\ sV_2 \\ sV_3 \end{bmatrix} = \begin{bmatrix} -11 & 1 & 0 \\ 10 & -11 & 1 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} V_0$$

Using Matlab to find the transfer function:

```
>> A = [-11,1,0 ; 10,-11,1 ; 0,10,-10]

   -11     1     0
    10   -11     1
     0    10   -10

>> B = [10;0;0];
>> C = [0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                    1000
G(s) = ----------------------------
       (s+15.25) (s+10.51) (s+6.24)
```

This is closer to having three poles at -10, but it's still a little off.