Butterworth and Chebuchev Filters

When designing a filter, you

- Place N poles near the region you want to pass, and
- Place zeros near the regions you want to reject.

Butterworth filters are a special case where you have

- N poles,
- No zeros, and
- The maximum gain must be less than 1.000 (no resonance)

Type-1 Chebychev filters loosen this up a little and have

- N poles,
- No zeros, and
- The maximum gain must be less than $1 + \varepsilon$ (a slight resonance)

Butterworth Filters

As it turns out, the optimal filter (i.e. the closest to an ideal low-pass filter) with

- N poles
- No zeros, and
- No resonance (max gain <= 1.0000)

has N poles with

- The amplitude of the poles is equal to the corner frequency, and
- The angle between the poles is

$$\phi = \frac{180^0}{N}$$

If the corner is 1 rad/sec, for example, the poles for a Butterworth filter are

N	2	3	4	5
poles	$-1\angle\pm 45^0$	-1 $-1\angle\pm 60^0$	$-1\angle\pm 22.5^0$ $-1\angle\pm 67.5^0$	-1 $-1\angle\pm 36^0$ $-1\angle\pm 72^0$

Example: Design a 4th-order Butterworth low-pass filter with a corner at 10 rad/sec.

Solution: The poles are 45 degrees apart

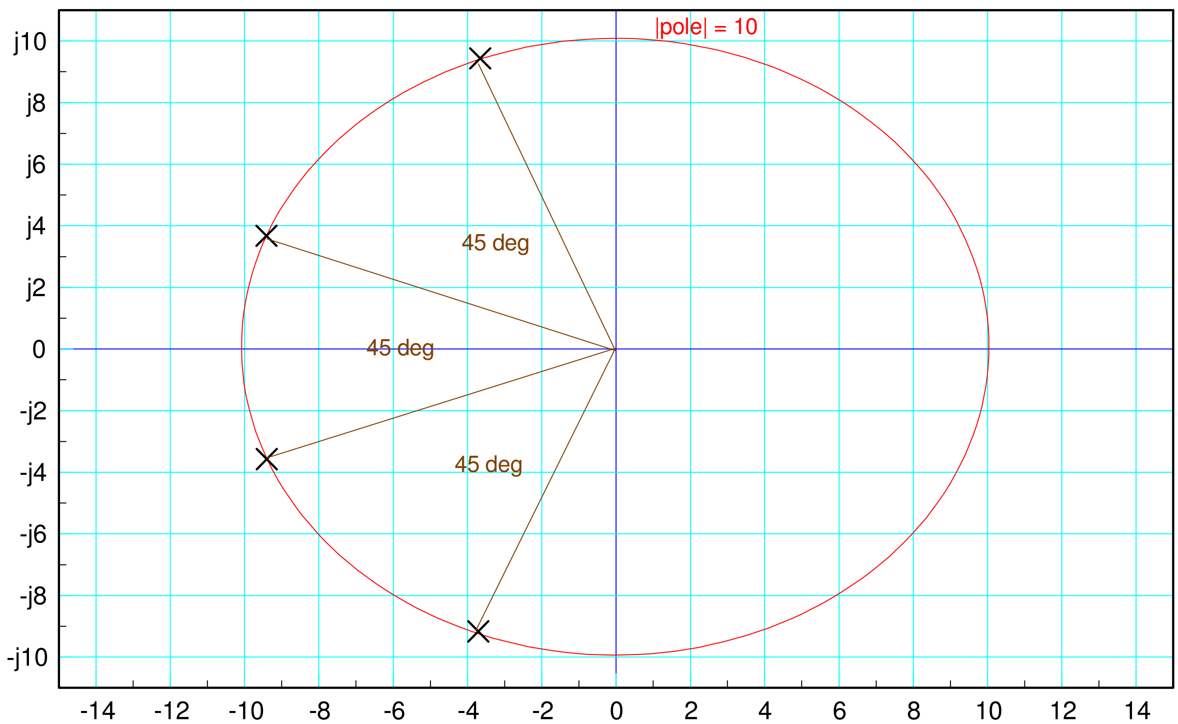
$$\phi = \frac{180^0}{4} = 45^0$$

The amplitude of the poles is the corner frequency (10 rad/sec). Hence

$$G(s) = \left(\frac{10^4}{\left(s+10\angle 22.5^0\right)\left(s+10\angle -22.5^0\right)\left(s+10\angle 67.5^0\right)\left(s+10\angle -67.5^0\right)} \right)$$

or since complex poles are always in complex conjugate pairs,

$$G(s) = \left(\frac{10^4}{\left(s+10\angle \pm 22.5^0\right)\left(s+10\angle \pm 67.5^0\right)} \right)$$



To design a 4th-order Butterworth low-pass filter, place four poles with amplitudes of 10, each 45 degrees apart

The resulting gain vs. frequency is then

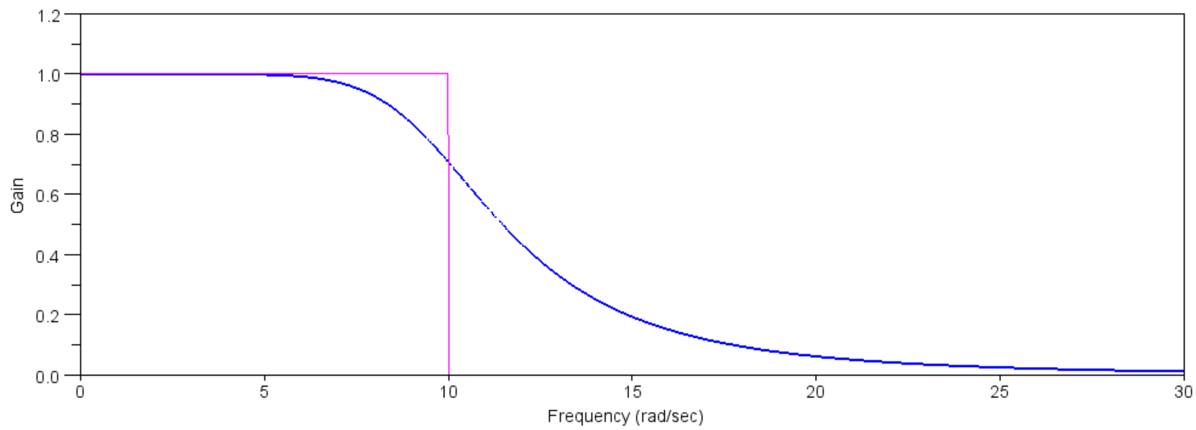
```
p1 = 10 * exp(j*22.5*pi/180);
p2 = conj(p1);
p3 = 10 * exp(j*67.5*pi/180);
p4 = conj(p3);

w = [0:0.01:30]';
s = j*w;

Gd = 1 * (w < 10);

Gs = 10^4  ./ ( (s+p1) .* (s+p2) .* (s+p3) .* (s+p4) );

plot(w,abs(Gs),'b',w,abs(Gd),'m');
xlabel('Frequency (rad/sec)');
ylabel('Gain');
```



Gain vs. Frequency for a 4th-Order Butterworth Low-Pass Filter

Chebuchev Filters

A Chebuchev filter is similar to a Butterworth filter, except you allow the gain to be slightly larger than one at some point. In return, you get a filter which is closer to ideal.

Unlike a Butterworth filter, there are an infinite number of Chebychev filters: one for each resonance you allow. The poles for a Type-1 Chebychev filter with a corner at 1 rad/sec follow and 0.2 ripple are given below.

	N=2	N=3	N=4	N=5	N=6
zeros	none	none	none	none	none
poles	$-1.60\angle\pm 50.7^0$	-0.85 $-1.21\angle\pm 69.5^0$	$-0.72\angle\pm 38.5^0$ $-1.11\angle\pm 77.8^0$	-0.48 $-0.76\angle\pm 59.3^0$ $-1.06\angle\pm 82.0^0$	$-0.47\angle\pm 36.1^0$ $-0.81\angle\pm 69.8^0$ $-1.04\angle\pm 84.4^0$

Example: Design a 4th-order Chebuchev filter with

- A DC gain of 1.000
- A ripple of 0.2, and
- A corner at 10 rad/sec

Solution:

$$G(s) = \left(\frac{7.2^2 \cdot 11.1^2}{\left(s+7.2\angle\pm 38.5^0\right)\left(s+11.1\angle\pm 77.8^0\right)} \right)$$

or

$$G(s) = \left(\frac{6387}{\left(s^2+11.27s+51.84\right)\left(s^2+4.69s+123.21\right)} \right)$$

Checking the gain vs. frequency in Matlab

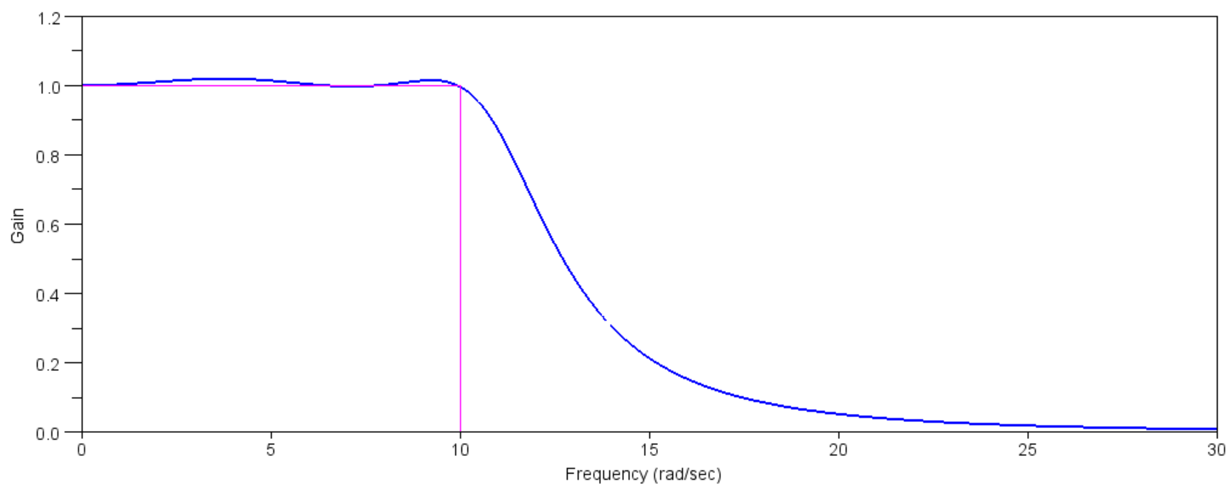
```
p1 = 7.2 * exp(j*38.5*pi/180);
p2 = conj(p1);
p3 = 11.1 * exp(j*77.8*pi/180);
p4 = conj(p3);

w = [0:0.01:30]';
s = j*w;

Gd = 1 * (w < 10) .* (w > -10);

Gs = 7.2^2 * 11.1^2  ./ ( (s+p1) .* (s+p2) .* (s+p3) .* (s+p4) );

plot(w,abs(Gs),'b',w,abs(Gd),'m');
xlabel('Frequency (rad/sec)');
ylabel('Gain');
```



Gain vs. Frequency for a 4th-Order Chebychev Filter with a Corner at 10 rad/sec