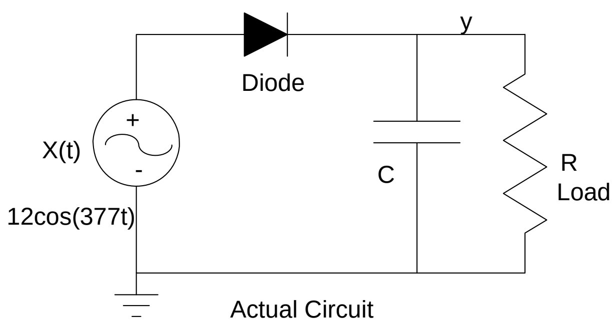
Circuit Analysis with LaPlace Transforms

Objective:

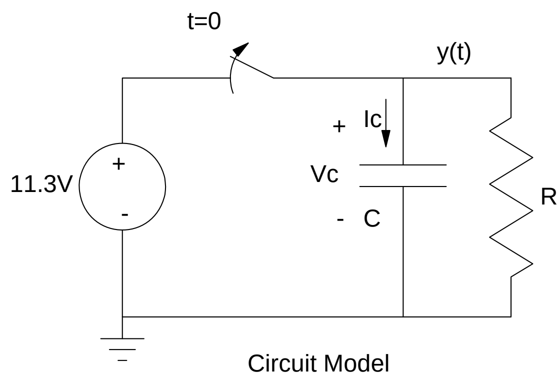
- Analyze RC and RL circuits with initial conditions

AC to DC Converter

The following ciruit on the left is a half-wave rectifier. It converts an AC signal to a DC signal. The input, x(t), is a 12V peak, 60Hz sine wave. The diode only turns on when the source voltage is greater than the load voltage (for an ideal diode). Assume you can drive 12V at 100mA (R = 120 Ohms) and C = 1,000uF,



Actual Circuit



Circuit Model

Numerical Solution: A numerical solution to Vc(t) can be fount using VISSIM, MATLAB, or SPICE.

Step 1: Write the differential equations that describe this system. Treat the diode as a resistor with resistance D

$$I_c + \left(\frac{y}{R}\right) + \left(\frac{y-x}{D}\right) = 0$$

$$C\frac{dy}{dt} + \frac{y}{R} + \frac{y-x}{D} = 0$$

$$\frac{dy}{dt} = -\left(\frac{1}{RC} + \frac{1}{DC}\right)y + \left(\frac{1}{DC}\right)x$$

Model the diode as a resistor which

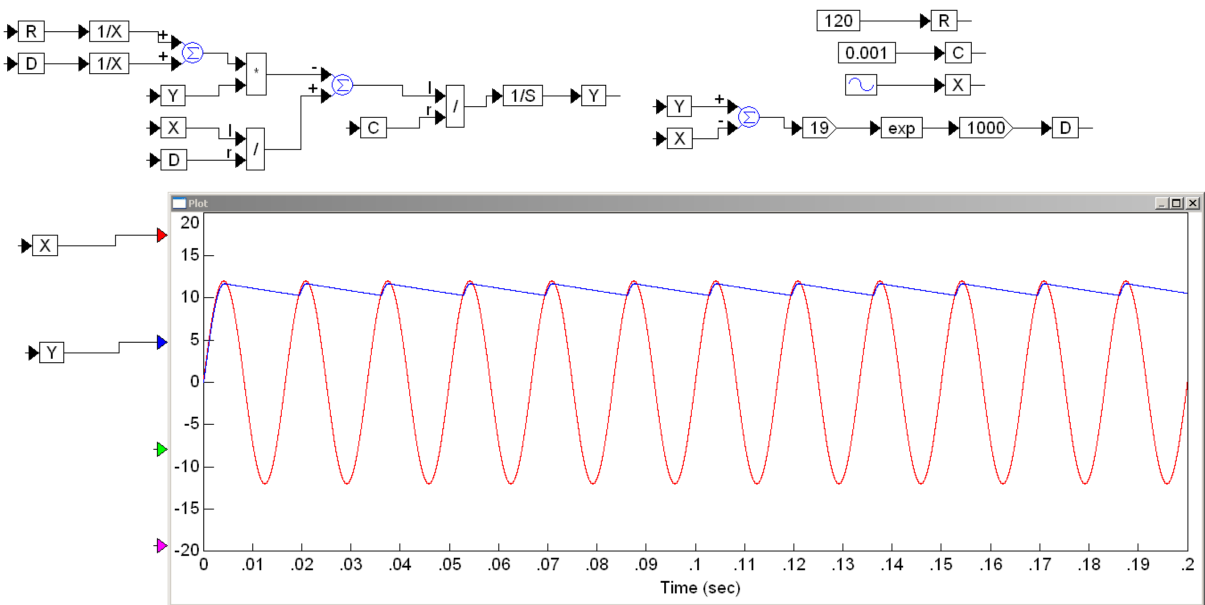
- has a large resistance when y > x
- has a low resistance when x > y

$$D \approx 10^7 e^{19(y-x)}$$

Fore example,

- X = 1V, Y = 0V, D = 0.05 Ohms (almost a short circuit)
- X = 0V, Y = 1V, D = 10^{15} Ohms (almost an open circuit)

Step 2: Input into VISSIM (or MATLAB, SciLab, SPICE, etc.)



Note that the output (the blue line) charges up when x(t) goes to +12V and then discharges until the next peak. The diode results in y(t) being slightly below x(t).

- If this were an ideal silicon diode, y(t) should be 0.7V below x(t).
- If this were an ideal germanium diode, y(t) should be 0.3V below x(t).

This model was adjusted so that the exponential was 19 $\left(\frac{1}{nV_T} = 19.23\right)$ and the contant out front gave a 0.7 volt drop.

Analytic Solution:

- Assume at t=0, x(t)=+12V, the diode is on, and the capacitor was charged up to 11.3V. (+12V minus the 0.7V across the diode).
- Assume for t>0, the diode is off.

This allows us to trace the shape of the blue curve as the diode discharges. This solution won't be valid past 1/60th of a second since x(t) is a sinusoid and will turn on again at that time. But, it lets us calculate the shape of the blue curve.

Recall:

$$\frac{dy}{dt} + \left(\frac{1}{RC} + \frac{1}{DC}\right)y = \left(\frac{1}{DC}\right)x$$

With the diode turned off:

$$\frac{dy}{dt} + \left(\frac{1}{RC}\right)y = 0$$

Step 2: Convert to LaPlace

$$(sY - y(0)) + \left(\frac{1}{RC}\right)Y = 0$$

Plugging in numbers:

$$(sY - 11.3) + (8.33)Y = 0$$

Step 3: Solve for Y

$$Y = \left(\frac{11.3}{s+8.33}\right)$$

Step 4: Convert back to time

$$y(t) = 11.3 \cdot e^{-8.33t}u(t)$$

The ripple is the bounce in the blue line. It is approximately the amount y(t) decays in 1/60th of a second

$$Ripple \approx (11.3) - (11.3 \cdot e^{-8.33t}u(t))_{t=1/60}$$

$$Ripple \approx 1.46V$$

If you want less ripple,

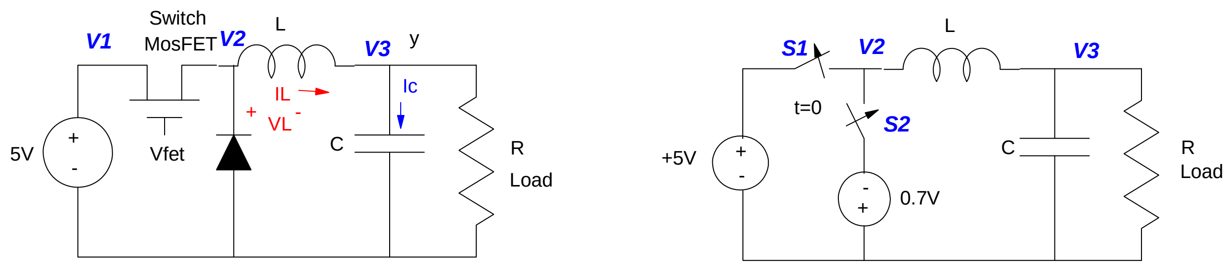
- Draw less current. That kind of defeats the idea of an AC to DC converter, though.
- Use a full-wave rectifier, so the capacitor discharges for 1/120th second rather than 1/60th of a second.
- Increase C. If you make C 10x larger, the ripple is 10x smaller (approximately).

Power supplies tend to have *huge* capacitors for this reason. The 5V power supply shown to the right (International Power IHN5-9/OVP from www.mpja.com) includes

- A transformer (the black cube on the left). This converts 120VAC to 5VAC.
- Electronics (the black circuit board to the top right - like described above - a little more complicated though to include over-voltage protection, etc.)
- A full-wave rectifier (cuts the ripple in half - but uses 4 diodes rather than the one used here)
- Capacitor (the big blue cyllinder in the lower right.) 2,000uF to 200,000uF are not uncommon for such power supplies.

Buck Converter

The following circuit convets +5VDC to 0..5VDC



Buck Converter Circuit (left) and model (right)

The MosFET acts as a switch that can be turned on and off.

- When turned on, +5V is applied to the inductor and the diode turns off. If you leave the switch on too long, the ouput will go to +5V.
- When turned off, the +5V source is disconnected from the circuit. In this case, the diode turns on to allow current to continue flowing in the industor. The energy in the inductor decays to zero according to some differential equation. The output, y(t), also decays to zero.
- By turning the MosFET on and off, you can regulate the output at any voltage between 0V and +5V. Presumably, there will be a microprocessor or comparitor monitoring the output voltage and turning the MosFET on and off appropriately.

Case 1: S1 closes at t=0 and S2 opens at t=0:

Prior to t=0, S1 was open and S2 was closed. The initial condition is then

$$v_3(0) = v_c = -0.7V$$

$$i_L(0) = \frac{v_3(0)}{R}$$

These set the initial conditions (needed for LaPlace transforms.)

Now let's assume S1 closes at t=0 and S2 opens. Taking the LaPlace transforms:

$$v_2 = 5$$ (LaPlace transform for a unit step of 5V)

$$L\frac{dI_L}{dt} = 5 - V_3$$

$$I_L = C\frac{dV_3}{dt} + \frac{V_3}{R}$$

Taking the LaPlace transform

$$L(sI_L - i(0)) = \left(\frac{5}{s}\right) - V_3$$

$$I_L = C(sV_3 - v_3(0)) + \frac{V_3}{R}$$

Grouping terms:

$$(Ls)I_L + (1)V_3 = \left(\frac{5}{s}\right) + Li(0)$$

$$(1)I_L - \left(Cs + \frac{1}{R}\right)V_3 = -Cv_3(0)$$

Solving for V3

$$\left(1 + Ls\left(Cs + \frac{1}{R}\right)\right)V_3 = \frac{5}{s} + Li(0) + LCsv_3(0)$$

$$s(RLCs^2 + Ls + R)V_3 = 5R + sRLi(0) + s^2RLCv_3(0)$$

Assume R = 10 Ohms, C =33.3uF, L = 15mH, v3(0) = -0.7V, i(0)=-70mA

$$V_3 = \left(\frac{-0.7s^2 - 30{,}000s + 10{,}000{,}000}{s^3 + 3{,}000s^2 + 2{,}000{,}000s}\right)$$

$$V_3 = \left(\frac{-0.7s^2 - 30{,}000s + 10{,}000{,}000}{s(s+1000)(s+2000)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+1000}\right) + \left(\frac{C}{s+2000}\right)$$

$$A = \left(\frac{-0.7s^2 - 30{,}000s + 10{,}000{,}000}{(s+1000)(s+2000)}\right)_{s=0} = 5$$

$$B = \left(\frac{-0.7s^2 - 30{,}000s + 10{,}000{,}000}{s(s+2000)}\right)_{s=-1000} = -39.3$$

$$C = \left(\frac{-0.7s^2 - 30{,}000s + 10{,}000{,}000}{s(s+1000)}\right)_{s=-2000} = 33.6$$

$$V_3 = \left(\frac{5}{s}\right) + \left(\frac{-39.3}{s+1000}\right) + \left(\frac{33.6}{s+2000}\right)$$

so

$$v_3(t) = (5 - 39.3e^{-1000t} + 33.6e^{-2000t})u(t)$$

Note that v3(t) charges up to +5V. At t=0, v3(t) was -0.7V.

Case 2: S1 opens at t=0 and S2 closes at t=0. Prior to t=0, S1 was closed and S2 was open. The initial conditions are then

$$v_3(0) = v_c = 5$$

$$i_L(0) = \frac{v_3(0)}{R}$$

For t>0,

$$v_2 = -0.7$$

$$L\frac{dI_L}{dt} = -0.7 - V_3$$

$$I_L = C\frac{dV_3}{dt} + \frac{V_3}{R}$$

Note that the equations are unchanged, except for the 5 goes to -0.7. This will result in

$$s(RLCs^2 + Ls + R)V_3 = -0.7R + sRLi(0) + s^2RLCv_3(0)$$

Plugging in R = 10 Ohms, C =33.3uF, L = 15mH, v3(0) = 5V, i(0)=500mA

$$V_3 = \left(\frac{5s^2+15{,}000s-1{,}400{,}000}{s(s+1000)(s+2000)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+1000}\right) + \left(\frac{C}{s+2000}\right)$$

$$A = \left(\frac{5s^2+15{,}000s-1{,}400{,}000}{(s+1000)(s+2000)}\right)_{s=0} = -0.7$$

$$B = \left(\frac{5s^2+15{,}000s-1{,}400{,}000}{s(s+2000)}\right)_{s=-1000} = 11.4$$

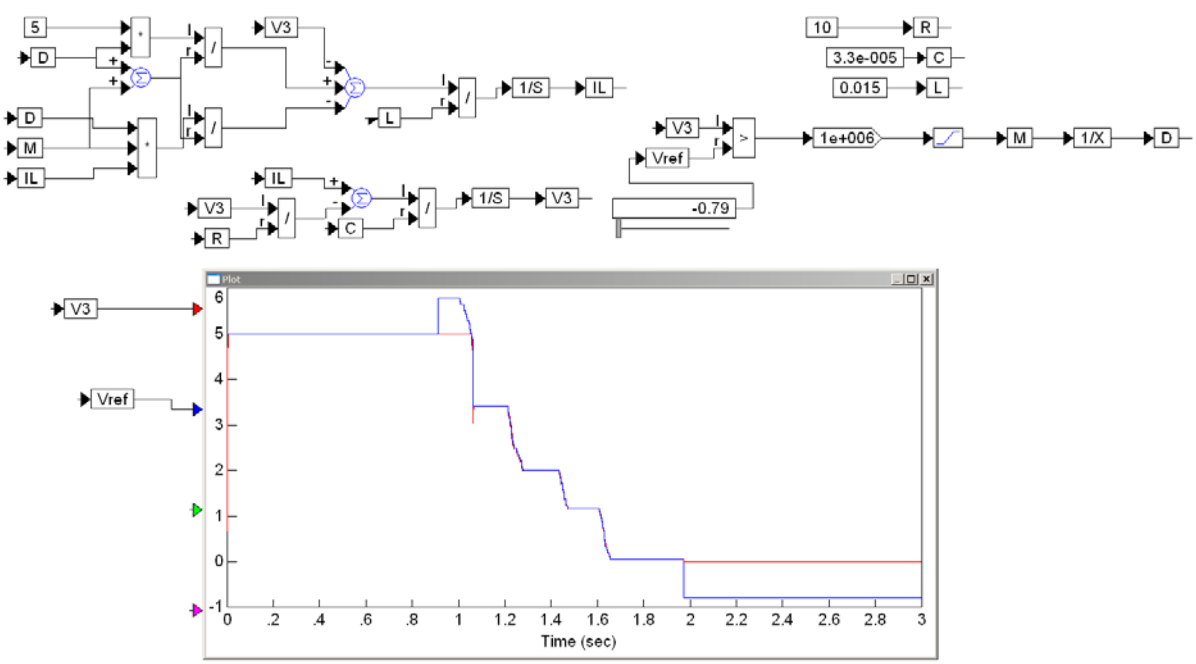
$$C = \left(\frac{5s^2+15{,}000s-1{,}400{,}000}{s(s+1000)}\right)_{s=-2000} = -5.7$$

$$V_3 = \left(\frac{-0.7}{s}\right) + \left(\frac{11.4}{s+1000}\right) + \left(\frac{-5.7}{s+2000}\right)$$

$$v_3(t) = (-0.7 + 11.4e^{-1000t} - 5.7e^{-2000t})u(t)$$

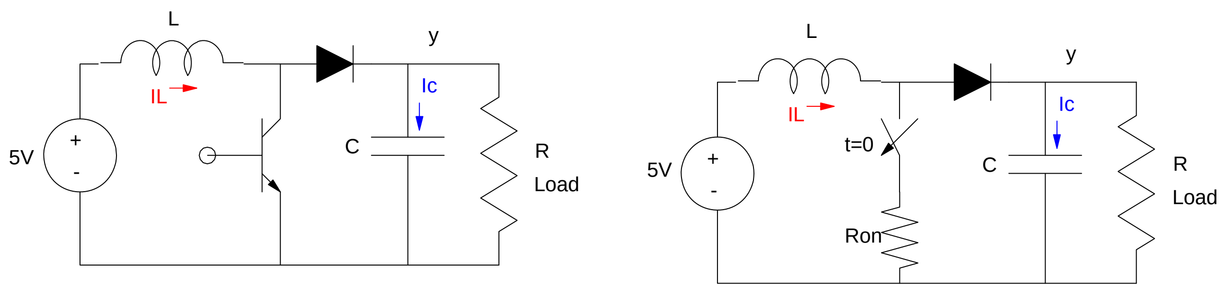
Note that v3(t) discharges down to -0.7V at t goes to infinity. At t=0, v3(t) = +5V.

A buck converter follows these two curves: the first one as you raise the output voltage, the second one as you drop the output voltage. If you switch quickly, you can hold the output at anything between 0V and 5V. (note: I'm not certain about the output below 0.7V - the voltage needed to turn on the diode. The math suggests you can go down to -0.7V - but that's based upon a model that might not be valid below 0.7V. I'd check this with a nonlinear simulation to make sure, though.) The VisSim simulation below shows the reference voltage (blue) and the output voltage (red) - indicating that the buck converter is tracking the reference for 0 < Vref < 5V.



VisSim simulation of a Buck converter

Boost Converter



A boost converter converts 5VDC to something higher than +5V. The idea is to use an inductor in parallel with a power supply with a switch (left figure above). When the switch is closed (the transistor saturates or a FET is placed in the low impedance region), the power supply produced current in the inductor. When the switch opens, the magnetic field in the inductor collapses, raising the voltage to whatever it takes to allow this energy to dissipate. This is how your spark plugs work (+12VDC in produces several thousand volts across the spark plug). You can also use this technique to produce more modest voltages, like +12VDC.

Each time you close and open the switch, more energy is sent to the capacitor as the inductor's field collapses. The more times you close the switch, the more charge (and higher voltage) you produce.

Problem: Find the equation which describes the current through the inductor as the switch is closed. Assume L = 100mH and R = 6 Ohms (the on resistance for a power FET).

1) Write the differential equations. Ignore the circuit to the right of the diode. The diode will be turned off so no current flows to the right.

$$L\frac{di_L}{dt} = 5 - i_L R_{on}$$

Taking the LaPlace transform

$$L(sI_L - i_L(0)) = \frac{5}{s} - I_L R_{on}$$

Solve for IL

$$s(Ls + R_{on})I_L = 5 + sLi_L(0)$$

$$I_L = \left(\frac{5}{s(Ls+R)}\right) + \left(\frac{Li_L(0)}{(Ls+R)}\right)$$

Assume

L = 100mH

Ron = 6 Ohms

Rload = 100 Ohms

Assume the switch was open for t<0. The steady-state solution is

$$i_L(0) = \left(\frac{5V-0.7V}{100\Omega}\right) = 43mA$$

$$I_L = \left(\frac{50}{s(s+60)}\right) + \left(\frac{0.043}{s+60}\right)$$

Taking the partial fraction expansion

$$I_L = \left(\frac{0.8333}{s}\right) + \left(\frac{-0.8333}{s+60}\right) + \left(\frac{0.043}{s+60}\right)$$

so

$$i_L(t) = (0.8333 - 0.7903e^{-60t})u(t)$$

Problem: Find the equation which describes the current through the inductor as the switch is opened. Assume L = 100mH, C = 100uF, and R = 100 Ohms. Assume the switch was closed for t<0.

Solution: Write the differential equation for this circuit with the switch opened. Assume the diode is on with a 0.7V drop across it.

$$L\frac{di_L}{dt} = 5 - 0.7 - v_c$$

$$C\frac{dv_c}{dt} = i_L - \frac{v_C}{R}$$

Taking the LaPlace transforms

$$L(sI_L - i_L(0)) = \left(\frac{4.3}{s}\right) - V_c$$

$$C(sV_c - v_c(0)) = I_L - \frac{V_C}{R}$$

Grouping terms:

$$(Ls)I_L + (1)V_c = \left(\frac{4.3}{s}\right) + Li_L(0)$$

$$(-1)I_L + \left(Cs + \frac{1}{R}\right)V_c = Cv_c(0)$$

Solving for Vc

$$\left(1+LCs^2+\frac{Ls}{R}\right)V_c=\left(\frac{4.3}{s}\right)+Li_L(0)+LCsv_c(0)$$

$$s(R+RLCs^2+Ls)V_c=(4.3R)+RLsi_L(0)+RLCs^2v_c(0)$$

The initial conditions assuming the switch was closed for t<0 is:

$$v_c(0)=4.3V$$

$$i_L(0)=\frac{5V}{6\Omega}=833.3mA$$

Plugging in numbers

$$s(0.001s^2+0.1s+100)V_c=0.0043s^2+8.3333s+430$$

$$V_c=\left(\frac{4.3s^2+8,333.3s+430,000}{s(s^2+100s+100,000}\right)=\left(\frac{4.3s^2+8,333.3s+430,000}{s(s+50+j312.25)(s+50-j312.25)}\right)$$

Using partial fractions

$$V_c=\left(\frac{A}{s}\right)+\left(\frac{B}{s+50+j312.25}\right)+\left(\frac{C}{s+50-j312.25}\right)$$

$$A=\left(\frac{4.3s^2+8,333.3s+430,000}{(s+50+j312.25)(s+50-j312.25)}\right)_{s=0}=4.3$$

$$B=\left(\frac{4.3s^2+8,333.3s+430,000}{s(s+50-j312.25)}\right)_{s=-50-j312.25}=12.65\angle 90^0$$

$$C=\left(\frac{4.3s^2+8,333.3s+430,000}{s(s+50+j312.25)}\right)_{s=-50+j312.25}=12.65\angle-90^0$$

$$V_c=\left(\frac{4.3}{s}\right)+\left(\frac{12.65\angle 90^0}{s+50+j312.25}\right)+\left(\frac{12.65\angle-90^0}{s+50-j312.25}\right)$$

so

$$v_c(t)=(4.3+25.31e^{-50t}\cos(312.25t-90^0))u(t)$$

Sidelight:

It looks like vc(t) starts at 4.3V and ends at 4.3V. The diode actually stops the capacitor from discharging, so this model stops when vc(t) is a maximum (at cos(0)):

$$312.25t-90^0=0^0$$

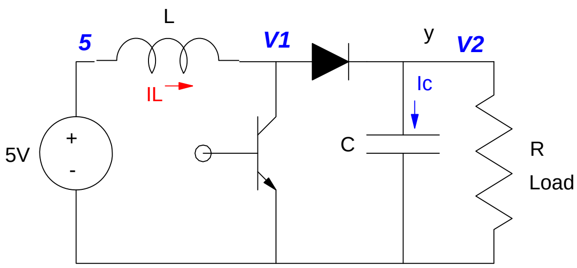
$$312.25t-\frac{\pi}{2}=0$$

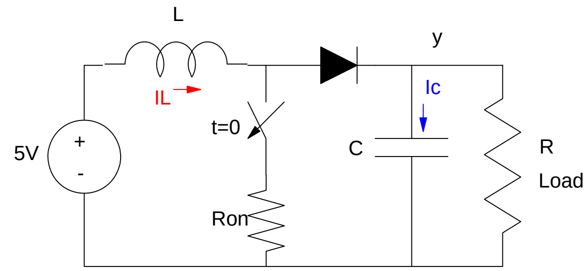
$$t=5ms$$

$$v_c(5ms)=23.98V$$

Vc is pumped up to 23.98V, at which point the diode turns off. The next time the switch is turned on and off, the initial condition on vc(0) is 23.98V (minus a little as the capacitor discharges through the resistive load). This pumps vc(t) higher. The diode then turns off again and the next time you close and open the switch, vc(t) is pumped even higher.

Nonlinear Simulation:

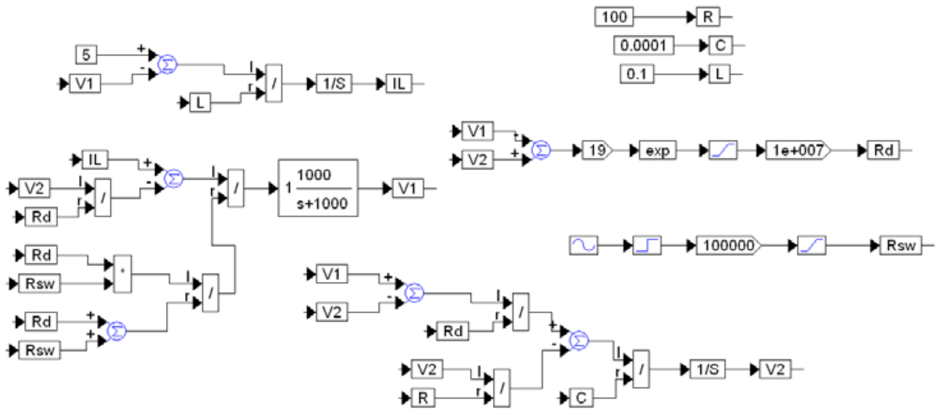


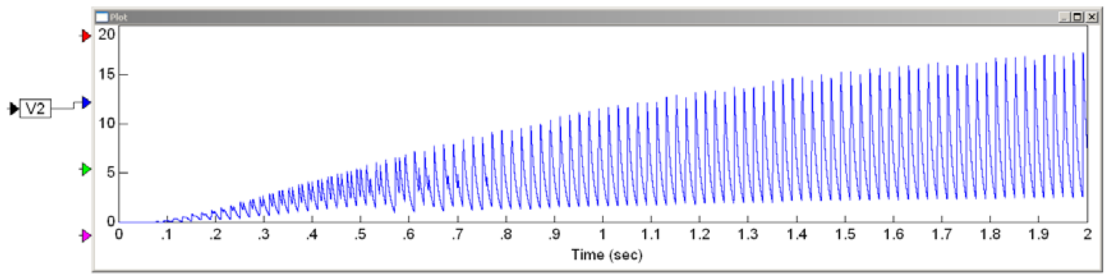


$$L\frac{di}{dt} = 5 - V_1$$

$$I_L = \frac{V_1}{R_{sw}} + \frac{V_1 - V_2}{R_D}$$

$$C\frac{dv_2}{dt} = \frac{V_1 - V_2}{R_D} + \frac{0 - V_2}{R}$$





Note that the capacitor is being charged when the switch opens and discharged when the diode turns off. The solutions above give the voltage vs. time for each portion. Their comnination and how it is pumping the output voltage up is something you almost need to use a numerical solution to find.