Fourier Transforms

Phasors and Superposition

Using phasors, you can solve for voltages and/or currents in a circuit with a sinusoidal input. The phasor representation for a voltage source is

$$v(t) = a\cos(\omega t) + b\sin(\omega t) \qquad \textit{time domain}$$

$$V = a - jb \qquad \textit{phasor (frequency) domain}$$

The impedance of resistors, capacitors, and inductors likewise become

$$Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = \frac{1}{j\omega C}$$

Standard circuit analysis techniques, such as voltage nodes, current loops, voltage division, etc. can then be used to analyze the circuit, albeit with complex numbers.

If an input is composed of several sinusoidal inputs, the currents and voltages can be found using superposition. Using superposition,

- The circuit is analyzed separately for each sinusoidal input .
- The total input is found by summing up each of the sinusoidal inputs.
- The total output is likewise found by summing up each of the resulting sinusoidal outputs.

If a circuit has an input which is periodic but *not* an explicit sum of sinusoids, Fourier Transforms are used to convert this signal to one which *is* an explicit sum of sinusoids. Then, phasor and superposition techniques can be used to solve for currents and voltages.

Fourier Transform

Assume a signal is periodic in time T:

$$x(t) = x(t+T)$$

The Fourier Transform for such a signal is

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}$$

What this says is, going right to left

If you add up a bunch of signals which are periodic in time T, the result is periodic in time T

This is a big *duh*. Going left to right, however, is much more interesting:

If you have a periodic signal which is not a pure sine wave, it is made up of harmonics.

To find the Fourier coefficients, one method is to note that all sine waves are orthogonal:

$$mean(\sin(at) \cdot \cos(bt)) = 0$$

$$mean(\sin(at) \cdot \sin(bt)) = \begin{cases} 0 & a \neq b \\ \frac{1}{2} & a = b \end{cases}$$

$$mean(\cos(at) \cdot \cos(bt)) = \begin{cases} 0 & a \neq b \\ \frac{1}{2} & a = b \end{cases}$$

With this, you can find the Fourier coefficients as:

$$a_0 = mean(x(t))$$ *a.k.a. the DC value of x(t)*

$$a_n = 2 \cdot mean(x(t) \cdot \cos(n\omega_0 t))$$ *cosine() terms*

$$b_n = 2 \cdot mean(x(t) \cdot \sin(n\omega_0 t))$$ *sine() terms*

Example: A good way to test any algorithm is to plug in a function where you know the answer. Assume a function which is periodic in 2π (meaning $\omega_0 = 1$)

$$x(t) = x(t + 2\pi)$$

$$x(t) = 1 + 3\cos(t) + 4\sin(2t)$$

In Matlab, you can determine the Fourier coefficients as follows:

```
t = [1:10000]' / 10000 * 2 * pi;
x = 1 + 3*cos(t) + 4*sin(2*t);
a0 = mean(x)
a0 =    1.0000

a1 = 2*mean(x .* cos(t))
a1 =    3.0000

b1 = 2*mean(x .* sin(t))
b1 =  2.9165e-015

a2 = 2*mean(x .* cos(2*t))
a2 = -3.0340e-015

b2 = 2*mean(x .* sin(2*t))
b2 =    4.0000

a3 = 2*mean(x .* cos(2*t))
a3 = -3.0340e-015

b3 = 2*mean(x .* cos(3*t))
b3 = 5.4526e-015
```

Note that each of the terms were picked out as expected.

harmonic	0	1	2	3	4	5	6
an	1	3	0	0	0	0	0
bn	0	0	4	0	0	0	0

Complex Fourier Transform

If you don't mind complex numbers, you can pull out both the sine() and cosine() terms with a single operation:

$$X_n = a_n - jb_n$$

$$X_n = 2 \cdot mean(x(t) \cdot e^{-jn\omega_0 t})$$

Example: Repeat the prevous case:

$$x(t) = x(t + 2\pi)$$

In Matlab:

```
X1 = 2*mean( x .* exp(-j*t) )
X1 =   3.0000 - 0.0000i

X2 = 2*mean( x .* exp(-j*2*t) )
X2 =  -0.0000 - 4.0000i

X3 = 2*mean( x .* exp(-j*3*t) )
X3 =  5.4526e-015 +5.7784e-016i
```

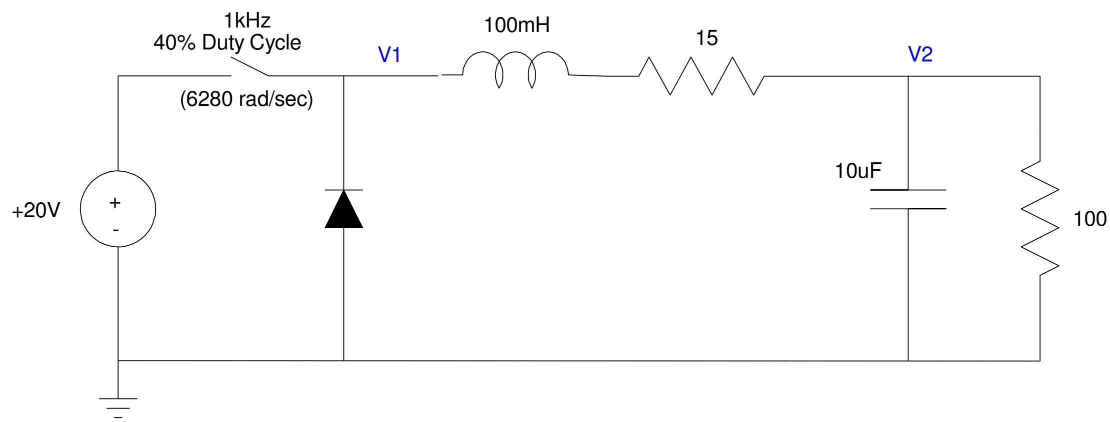
Note that this is the same answer as before, only using Phasor notation

harmonic	0	1	2	3	4	5	6
Xn	1	3 + j0	0 - j4	0	0	0	0

Once you can express x(t) in terms of sinusoids, you can analyze a circuit using superposition and phasor techniques.

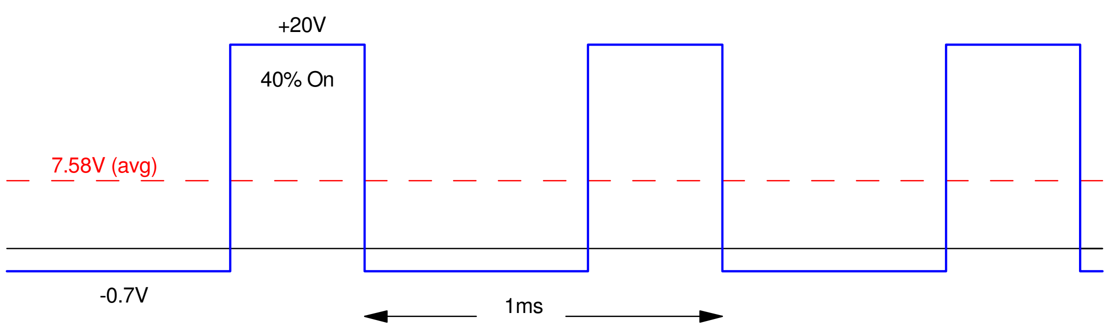
Circuit Analysis with Fourier Transforms: Buck Converter

Consider the Buck converter analyzed previously:



Buck Converter from Previous Lecture

The signal at V1 looks like the following:



Voltage at V1: When the switch closes, V1 = +20V (40% of the time).
When the switch opens, V1 = -0.7V due to the diode (60% of the time) - for an average voltage of 7.58V

Previously, we approximated this signal at V1 as

- A DC term (7.58V), and
- An AC term (20.7Vpp @ 1kHz)

The answers we got for the voltage at V2 were close to what Circutilab computed, but a little off. Using Fourier Transforms, you can get more accurate results.

Step 1: Find the Fourier Series expansion for V1(t). Note that time doesn't matter when finding the Fourier coefficients. For convenience, define

- V1(t) = x(t)
- Let the period be 2π (making $\omega_0 = 1$)

In Matlab:

```
t = [1:10000]' / 10000;
x = 20*( t < 0.4)  - 0.7*( t >= 0.4);
t = t * 2 * pi;
X0 = mean(x)

X0 =  7.5779

X1 = 2*mean( x .* exp(-j*t) )
X1 =   3.8725 -11.9184i

X2 = 2*mean( x .* exp(-j*2*t) )
X2 =  -3.1360 - 2.2784i

X3 = 2*mean( x .* exp(-j*3*t) )
X3 =   2.0861 - 1.5157i

X4 = 2*mean( x .* exp(-j*4*t) )
X4 =  -0.9686 - 2.9811i

X5 = 2*mean( x .* exp(-j*5*t) )
X5 =  -0.0041 + 0.0000i
```

What this means is that

```
V1(t) = 7.5779
      + 2.8725 cos(t) + 11.9184 sin(t)
      - 3.1360 cos(2t) + 2.2784 sin(2t)
      + 2.0861 cos(3t) + 1.515 sin(3t)
      - 0.9686 cos(4t) + 2.9811 sin(4t)
      - 0.0041 cos(5t) + 0.0000 sin(5t)
```

Now you can use superposition to solve for V2(t)

- Treat this as 6 separate problems: each at a different frequency
- Solve for V2(t) at each frequency
- Add up the answers to get the total answer.

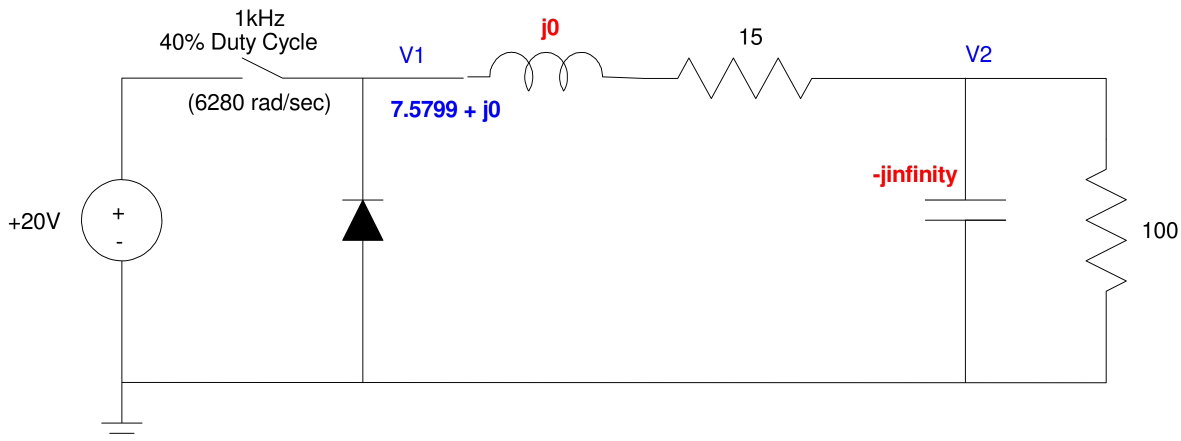
DC Analysis: $V_1(t) = 7.5779$

Redraw the circuit at w = 0 and solve for V2

$$V_1 = 7.5779 + j0$$

$$V_2 = \left(\frac{100}{100+15}\right) 7.5779$$

$$V_2 = 6.5895$$



1st Harmonic: 1000Hz

$$V_1 = 2.8725\cos(\omega_0 t) + 11.9184\sin(\omega_0 t)$$

In phasor form

$$V_1 = 2.8725 - j11.9184$$

$$\omega_0 = 2\pi \cdot 1000 \ \frac{rad}{\sec}$$

Redraw the circuit at 1000Hz and solve for V2

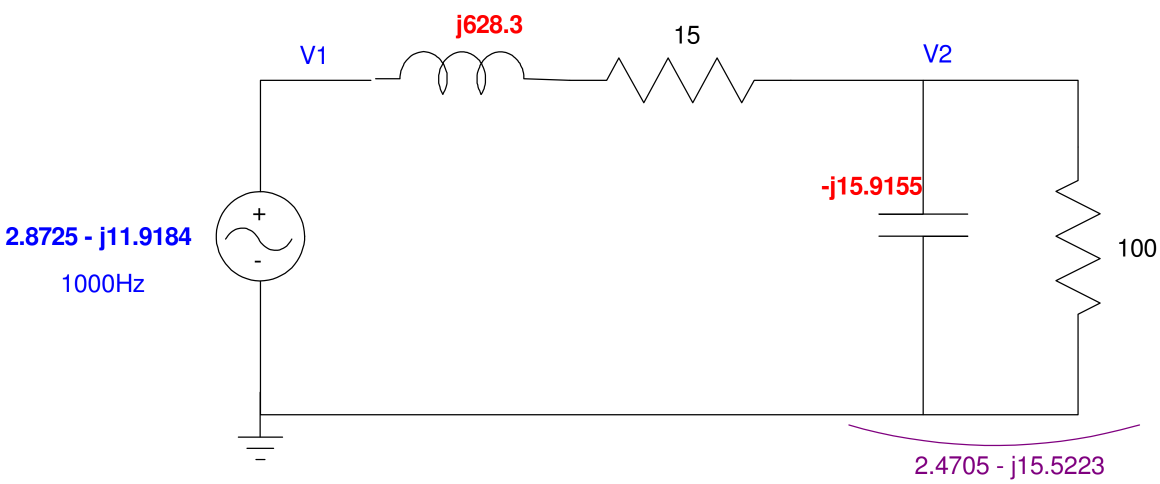
$$100\Omega \,||\, -j15.9155\Omega = 2.4705 - j15.5223$$

$$V_2 = \left(\frac{(2.4705-j15.5223)}{(2.4705-j15.5223)+(15+j628)}\right)(2.8725 - j11.9184)$$

$$V_2 = -0.1290 + j0.2866$$

Converting back to time domain

$$V_2(t) = -0.1290\cos(\omega_0 t) - 0.2866\sin(\omega_0 t)$$



2nd Harmonic (2000Hz)

$$V_1 = -3.1360\cos(2\omega_0 t) + 2.2784\sin(2\omega_0 t)$$

Convert to phasor form and analyze the circuit at 2000Hz

$$\omega = 2\omega_0 = 2000Hz = 12,566\tfrac{rad}{\sec}$$

$$V_1 = -3.1360 - j2.2784$$

$$L \to j\omega L = j1256.6\Omega$$

$$C \to \tfrac{1}{j\omega C} = -j7.9577\Omega$$

Find V2

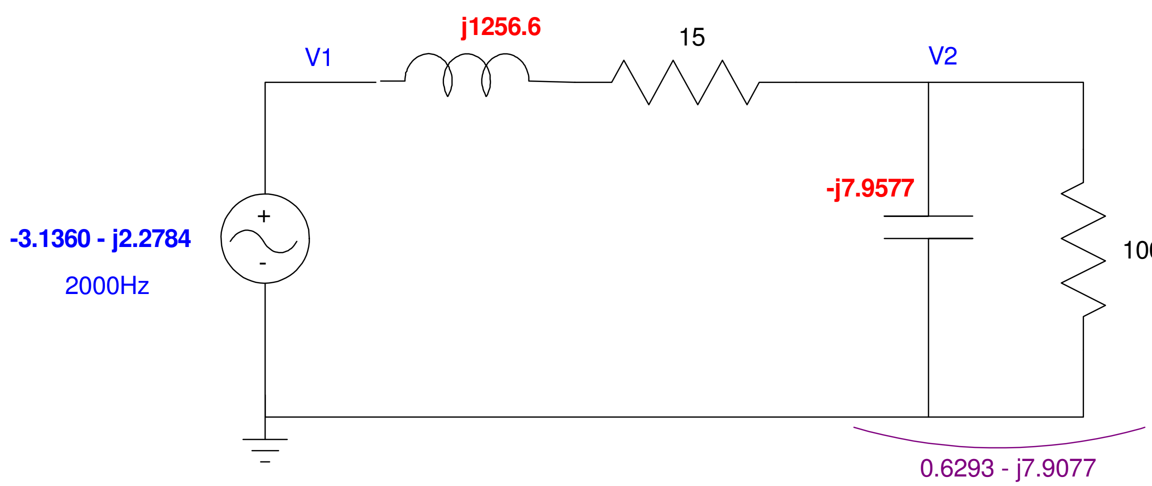
$$100 \,||\, -j7.9577 = 0.6293 - j7.9077$$

$$V_2 = \left(\tfrac{(0.6293 - j7.9077)}{(0.6293 - j7.9077) + (15 + j1256.6)}\right)(-3.1360 - j2.2784)$$

$$V_2 = 0.0185 + j0.0162$$

meaning

$$V_2(t) = 0.0185\cos(2\omega_0 t) - 0.0162\sin(2\omega_0 t)$$



3rd Harmonic: 3000Hz

$$V_1 = 2.0861\cos(3\omega_0 t) + 1.5150\sin(3\omega_0 t)$$

Convert to phasor form and analyze at 3000Hz

$$\omega = 3\omega_0 = 18,849\frac{rad}{\sec}$$

$$V_1 \rightarrow 2.0861 - j1.5150$$

$$L \rightarrow j\omega L = j1,884.9\Omega$$

$$C \rightarrow \frac{1}{j\omega C} = -j5.3052\Omega$$

Solve for V2

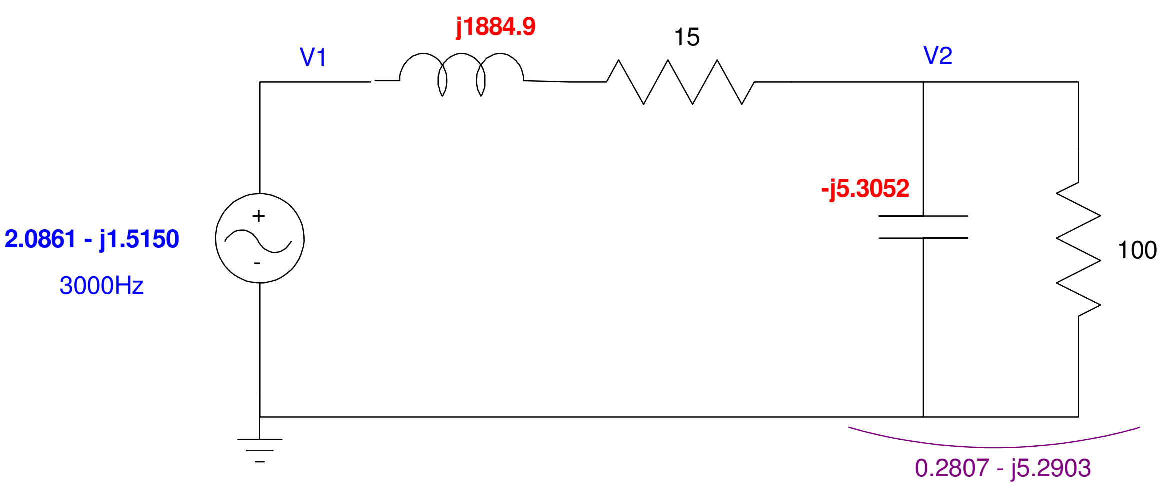
$$100 \,||\, -j5.3052 = 0.2807 - j5.2903$$

$$V_2 = \left(\frac{(0.2807-j5.2903)}{(0.2807-j5.2903)+(15+j1884.9)}\right)(2.0861 - j1.5150)$$

$$V_2 = -0.00061 + j0.00039$$

meaning

$$V_2(t) = -0.00061\cos(3\omega_0 t) - 0.00039\sin(3\omega_0 t)$$



4th Harmonics: 4000 Hz

$$V_1 = -0.9684\cos(4\omega_0 t) + 2.9811\sin(4\omega_0 t)$$

Convert to phasor form and analyze at 3000Hz

$$\omega = 4\omega_0 = 25,132.7\tfrac{rad}{\sec}$$

$$V_1 \rightarrow -0.9684 - j2.9811$$

$$L \rightarrow j\omega L = j2,513.27\Omega$$

$$C \rightarrow \tfrac{1}{j\omega C} = -j3.9789\Omega$$

Solve for V2

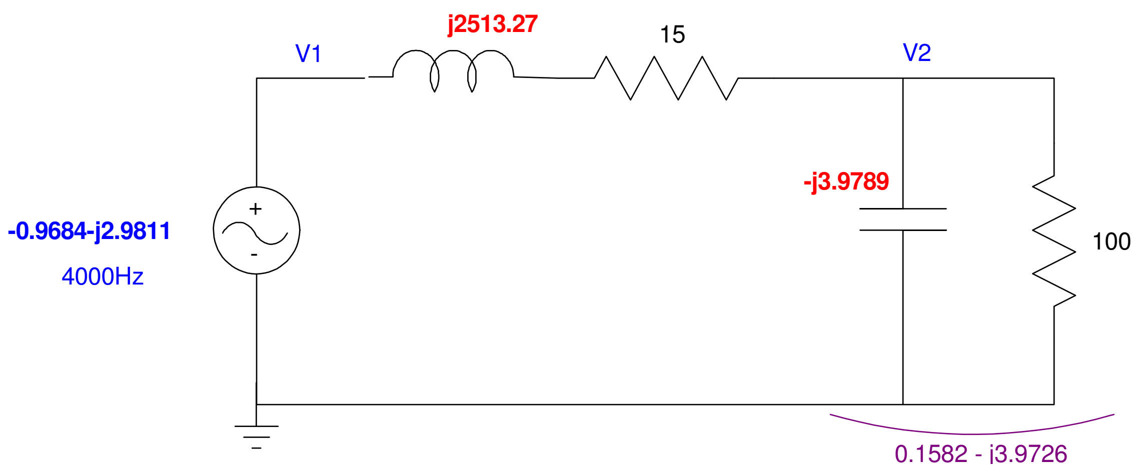
$$100\ ||\ -j3.9798 = 0.1582 - j3.9726$$

$$V_2 = \left(\frac{(0.1582-j3.9726)}{(0.1582-j3.9726)+(15+j2513.27)}\right)(-0.9684 - j2.9811)$$

$$V_2 = 0.00132 + j0.00479$$

meaning

$$V_2(t) = 0.00132\cos(4\omega_0 t) - 0.00479\sin(4\omega_0 t)$$

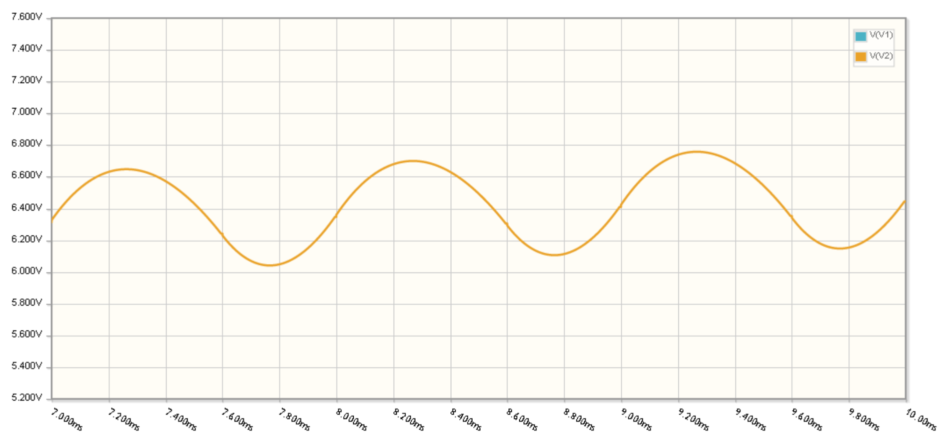


Putting it all together, V2(t) is

$$
\begin{aligned}
V_2 = 6.5895 + \\
&-0.1290 \cos(\omega_0 t) - 0.2866 \sin(\omega_0 t) \\
&+0.0185 \cos(2\omega_0 t) - 0.0162 \sin(2\omega_0 t) \\
&-0.00061 \cos(3\omega_0 t) - 0.00039 \sin(3\omega_0 t) \\
&+0.00132 \cos(4\omega_0 t) - 0.00479 \sin(4\omega_0 t) \\
&+\cdots
\end{aligned}
$$

This actually matches up with what CircuitLab predicts for this circuit. Note the following

- In theory, you need to include an infinite number of terms to represent V2(t). In practice, the terms quickly go to zero so you only need to include a few.
- Fourier Transforms allow you to compute the explicit form for V2(t),
- Fourier Transforms are more accurate than what we did last lecture, and
- They are a ***lot*** more work.



CircuitLab simulation for V2(t)

	V2(DC)	V2(AC)
Calculated Lecture 14	6.5895 V	530.5 mVpp
Calculated Fourier Transform	6.5895 V	628.5 mVpp
Simulated	6.404 V	592 mVpp