Complex Numbers

Objective:

- Become familiar with using complex numbers for addition, subtraction, multiplication, and dificion

Complex Numbers:

One of the greatest inventions in mathematics was the concept of the number zero. With the number zero, you can express the absence of something. It allows us to use a decimal number system rather than roman numerals. For example, try adding the following numbers using Roman numerals:

MXXIII + CVI =

If this is hard, try adding the following numbers:

1023 + 106 =

Zero allows you to use decimal numbers where it is possible to have numbers like 106 (zero tens).

Another great invention was negative numbers. This allows you to treat profit and loss alike mathematically - the only difference is the sign. Before negative numbers were invented, accountants had to represent loss as a positive debit, profit as a positive credit, and try to keep track of which column each entry belonged. The doule-enty bookkeeping system was significant enough that a small country, like Holland, was able to compete against large contries, like France, simply because they were better at keeping track of what ventures were profitable and which were not. It also resulted in England asking Holland to take over their govenrmnent (hence the German kings of England.)

Complex numbers allow you to respresent sines and cosines with a single number. This allows you to solve a differential equation with sinusoidal inputs by solving a single equation rather than two equations. The catch is the numbers you're using are complex.

Two basic definitions for complex numbers are

$$j^2 = -1$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Polar and Rectangular Form:

A complex number can be represented in rectangular form:

$$a + jb$$

or polar form

$$c \cdot e^{j\theta} = c\angle\theta$$

The relationship is

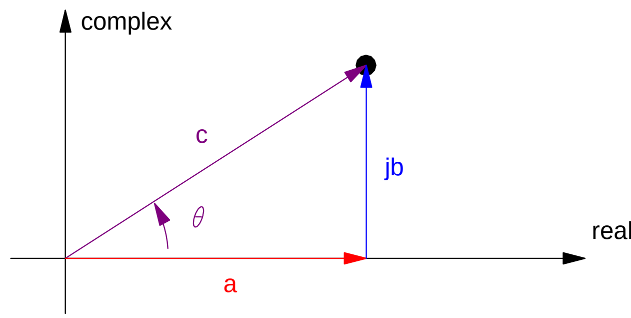
$$c = \sqrt{a^2 + b^2}$$

$$\tan(\theta) = \frac{b}{a}$$

or

$$a = c \cdot \cos(\theta)$$

$$b = c \cdot \sin(\theta)$$

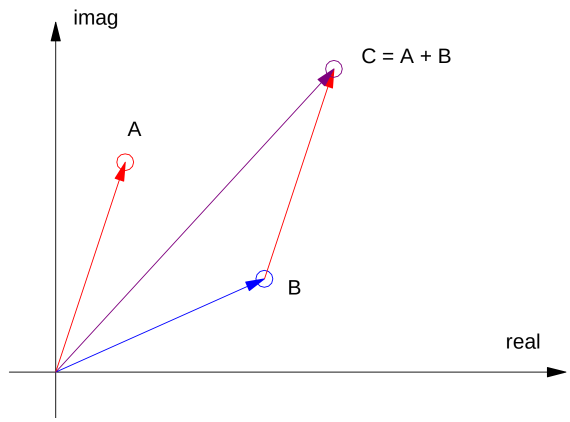


Addition & Subtraction of Complex Numbers

Rectangular for works best here. Let

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

To add complex numbers, add the reals and the imaginary parts seperately.



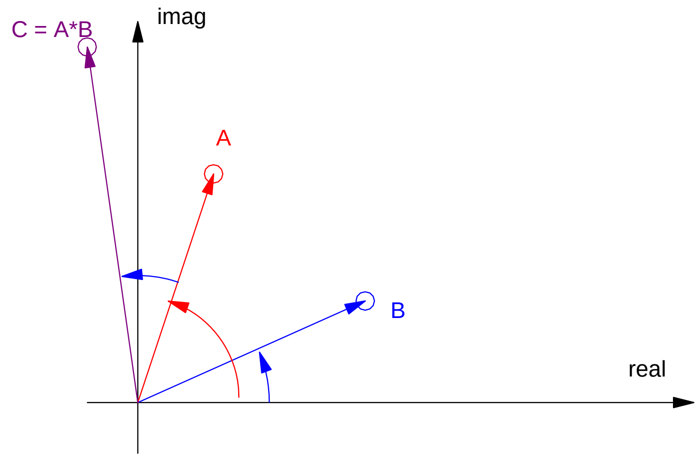
Multiplication and Division of Complex Numbers:

Polar form works best here:

$$(a\angle\theta)(b\angle\phi) = (ab)\angle(\theta+\phi)$$

The amplitude is the product of the two amplitudes.

The angle is the sum of the two angles



Most engineering calculators will allow you to use complex numbers the same as real numbers. If your calculator doesn't do this, consider buying a new calculator (my recommendation is an HP35s)

More Fun with Complex Numbers:

You can also do functions of complex numers.

$$y = \ln(1+j2)$$

$$= \ln(2.236\angle 63.4^0)$$

$$= \ln(e^{0.8047}\cdot e^{1.10 rad})$$

$$= 0.8047 + j1.10$$

Note that the natural unit for angle is radians. Pretty much anything English isn't natural.

Example 2:

$$y = \cos(2 + j3)$$

Note that

$$\cos(x) = \left(\frac{e^{jx} - e^{-jx}}{2} \right)$$

$$\cos(2 + j3) = \left(\frac{e^{j(2+j3)} + e^{-j(2+j3)}}{2} \right)$$

$$= \left(\frac{1}{2} \right) (e^{j2} e^{-3} + e^{-j2} e^{3})$$

$$= \left(\frac{e^{-3}}{2} \angle 2 \right) + \left(\frac{e^{3}}{2} \angle -2 \right)$$

$$= (-0.0104 + j0.0226) + (-4.179 - j9.132)$$

$$\boxed{\cos(2 + j3) = -4.1894 - j9.1094}$$

Example 3:

$$(1 + j2)^{(3+j4)}$$

$$= (2.236 \angle 63.43^0)^{(3+j4)}$$

$$= (e^{0.8047} \cdot e^{j1.1071})^{(3+j4)}$$

$$= (e^{(0.8047 + j1.1071)})^{(3+j4)}$$

$$= e^{(0.8047 + j1.1071)(3+j4)}$$

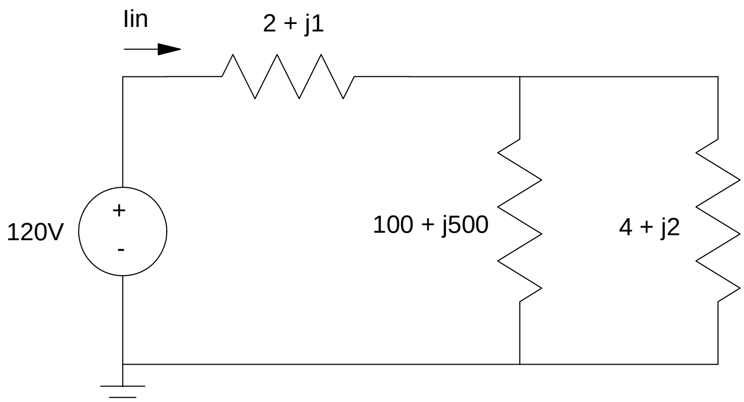
$$= e^{(-2.0143 + j6.5401)}$$

$$= (e^{-2.0143})(e^{j6.5401})$$

$$= 0.1334 \angle 6.5401 rad$$

$$\boxed{(1 + j2)^{(3+j4)} = 0.1290 + j0.0339}$$

Example: Determine the current, Iin:



Solution:

Resistors in parallel add as the sum of the inverses inverted:

$$\left(\left(\frac{1}{100+j500}\right)+\left(\frac{1}{4+j2}\right)\right)^{-1} = 3.9647 + j2.0166$$

Resistors in series add

$$(2+j1)+(3.9647+j2.0166) = 5.9647 + j3.0166$$

From V = IR

$$I = \frac{V}{R}$$

$$I = \frac{120V}{5.9647+j3.0166}$$

$$I = 16.0207 - j8.1023$$

or if you prefer polar form

$$I = 17.953\angle -26.82^0$$

Problem: Determine the voltage across the (4+j2) Ohm resistor

Solution: Use voltage division

$$V = \left(\frac{R_1}{R_1+R_2}\right)V_{in}$$

$$V = \left(\frac{(3.9647+j2.0166)}{(3.9647+j2.0166)+(2+j1)}\right)120V$$

$$V = 78.8563 + j0.1840$$

or if you prefer polar form

$$V = 78.8565\angle 0.1320^0$$