Phasors

Objective:

- Find the forced response to a differential equation with a sinusoidal input using phasor analysis
- Solve AC circuits using techniques from EE206 using phasor techniques

The goal of phasor analysis is to use all the techniques from EE206 for DC analysis with AC circuits which include inductors and capacitors. To do this, we need to use complex numbers and phasors. With this tool, all previous techniques, like current loops, voltage notes, etc work. The only catch is your coefficients are complex numbers.

Phasor Representation of a Sinusoid:

A sinusoidal signal has three degrees of freedom: the amplitude, frequency, and phase shift:

$$y(t) = a \cdot \cos(\omega t + \theta)$$

Assuming the frequency is known and fixed, there are only two degrees of freedom: the amplitude and phase shift. Phasor notation represents this as a complex number:

$$a \cdot \cos(\omega t + \theta) \equiv a\angle\theta$$

Another way to look at this uses Euler's identity:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

With a complex exponential, you can create a cos() or sin() function by taking the real part:

$$\cos(\omega t) = real(e^{j\omega t})$$

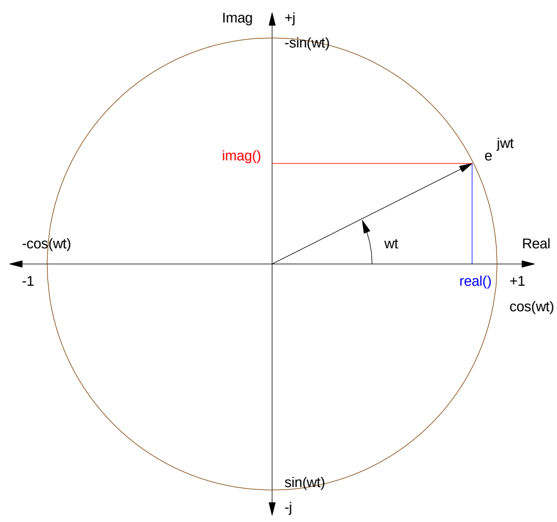
$$\sin(\omega t) = real(-je^{j\omega t})$$

You can thus represent cos() and sin() with the coefficient:

$$\cos(\omega t) \Leftrightarrow 1\angle 0^0$$

$$\sin(\omega t) \Leftrightarrow -j = 1\angle -90^0$$

The relatioship between sin() and cos() can be seen on the following phasor diagram:



Phasor Diagram: +1 = cos(), -j = sin()

Solving Differential Equations Using Phasors:

One use of phasors is to solve differential equations with sinusoidal inputs. For example, find y(t):

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 4\cos(3t)$$

Solution:

Step 1: Assume all functions are in the form of

$$y(t) = a \cdot e^{st}$$

Differentiation then becomes multiplication by 's'

$$\frac{dy}{dt} = s \cdot ae^{st} = sy$$

The LaPlace operator 's' thus translates to 'the derivative of'.

Step 2: Replace the derivatives with 's'

$$s^2Y + 6sY + 10Y = 4\cos(3t)$$

Solve for Y

$$Y = \left(\frac{4}{s^2+6s+10}\right) \cdot \cos(3t)$$

The gain from the input to y() is $\left(\frac{4}{s^2+6s+10}\right)$ for all 's'. The only point you care about is $s = j3$:

$$\cos(3t) = real(1 \cdot e^{j3t}) = 1\angle 0^0$$

Step 3: Evaluate at $s = j\omega$

$$Y = \left(\frac{4}{s^2+6s+10}\right)_{s=j3} \cdot 1\angle 0^0$$

$$Y = (0.2219\angle -86.82^0)$$

which means

$$y(t) = 0.2219\cos(3t - 86.82^0)$$

Solving Circuits with Phasors:

Assume all signals are in the form of e^{st} in general, or $e^{j\omega t}$ for the special case of sinusoidal inputs. The impedance of a resistor is

$$v = R \cdot i$$

The impedance of a capacitor is

$$v = \frac{1}{C}\int(i)dt$$

If i(t) is in the form of e^{st}, then integration becomes division by s or $j\omega$

$$i(t) = e^{st}$$

$$\int(i)dt = \frac{1}{s}e^{st} = \frac{1}{s} \cdot i(t)$$

Then

$$V = \left(\frac{1}{Cs}\right)I = \left(\frac{1}{j\omega C}\right)I$$

The impedance of a capacitor is $Z_c = \left(\frac{1}{j\omega C}\right)$

The impedance of an inductor is

$$v = L\frac{di}{dt}$$

's' means derivative, so

$$V = (Ls)I$$

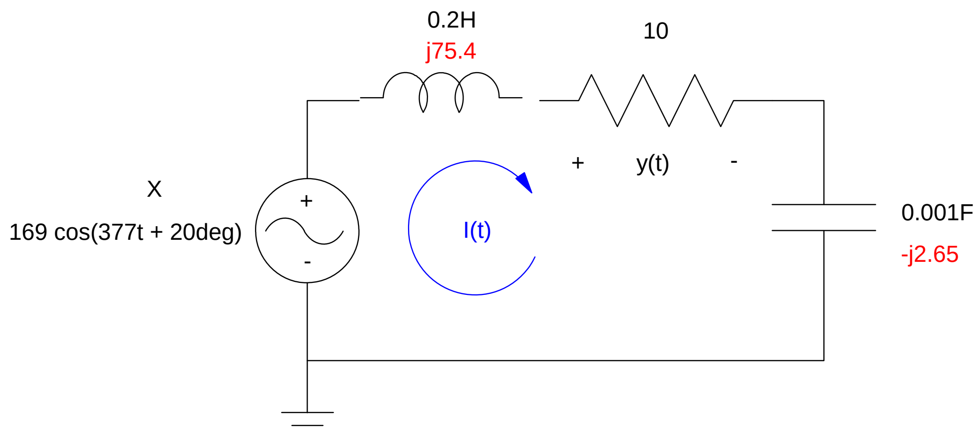
The impedance of an inductor is 'Ls' or $Z_L = j\omega L$

In summary:

Component	Imedance		
Input	LaPlace	Phasor	Phasor @ 60Hz
R	R	R	R
L	Ls	jwL	j377L
C	1 / Cs	1 / jwC	1 / j377C

Note that phasor analysis applied whenever you are solving a differential equation (or a circuit with capacitors and inductors) with a sinusoidal input. In this class, we are almost always dealing with 60Hz, however. Since the frequency is known and fixed, it is often used.

Example 1: Find the voltage across the 10 Ohm resistor, y(t):



First, convert to phaser notation. For this forcing function, x(t),

$$s = j377$$

Change the input to phaser notation:

$$X = 169\angle 20^0$$

Change RLC to phaser impedances:

$$R \to R = 10\Omega$$

$$L \to j\omega L = j75.4\Omega$$

$$C \to \frac{1}{j\omega C} = -j2.65\Omega$$

Now, use techniques from EE206 to find Y.

(a) Solve for the current:

$$-169\angle 20^0 + (j75.4 + 10 - j2.65)I = 0$$

$$(10 + j72.75)I = 169\angle 20^0$$

$$I = 2.30\angle -62.17^0$$

The output voltage (Y) is

$$Y = (10)I$$

$$Y = 23.01\angle -62.17^0$$

meaning

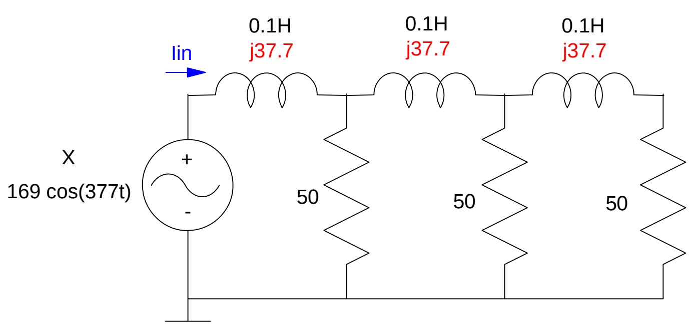
$$y(t) = 23.01\cos(377t - 62.17^0)$$

(b) You could also use voltage division:

$$Y = \left(\frac{10}{10+j75.4-j2.65}\right)\cdot 169\angle 20^0$$

$$Y = 23.01\angle -62.17^0$$

Example 2: Find the current Iin:



First, convert to phaser notation. In this case, due to the input being a 377rad/sec sine wave (60Hz):

$$s = j377$$

The phaser impedances are then:

0.1H = j37.7 Ohms

Simplify this circuit by adding impedances in series and parallel:

50 + j37.7 || 50 = 28.1111 + j8.2521

(28.1111 + j8.2521) + j37.7 = 28.1111 + j45.9521

(28.1111 + j45.9521) || 50 = 26.2232 + j13.9877

(26.2232 + j13.9877) + j37.7 = 26.2232 + j51.6877

The input sees an impedance of (26.2232 + j51.6877) Ohms. The current is then

$$I_{in} = \frac{X}{Z_{in}} = \frac{169\angle 0^0 V}{(26.2232+j51.6877)\Omega}$$

$$I_{in} = 1.3192 - j2.6003 \text{ Amps}$$

or if you prefer polar form:

$$I_{in} = 2.9158\angle -63.0996^0 \text{ Amps}$$