# ECE 341 - Homework #11

Markov Chains. Due Monday, June 7th

Please make the subject "ECE 341 HW#11" if submitting homework electronically to Jacob_Glower@yahoo.com (or on blackboard)

Problem 1 & 2) Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 50% chance of winning
- There is a 10% chance of a tie, and
- Team B has a 40% chance of winning

In order to win the match, a team must be up by 2 games.

1) Determine the probabilty that team A wins the match after k games for k = {0 ... 10} using matrix multiplication.

Let

$$X = \begin{bmatrix} \text{A is +2 (A wins)} \\ \text{A is +1} \\ \text{even} \\ \text{A is -1} \\ \text{A is -2 (B wins)} \end{bmatrix}$$

The state transistion matrix is then

$$X(k+1) = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0.5 & 0 & 0 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} X$$

```
>> A = [1,0.5,0,0,0 ; 0,0.1,0.5,0,0 ; 0,0.4,0.1,0.5,0 ; 0,0,0.4,0.1,0 ;
0,0,0,0.4,1]

    1.0000    0.5000         0         0         0
         0    0.1000    0.5000         0         0
         0    0.4000    0.1000    0.5000         0
         0         0    0.4000    0.1000         0
         0         0         0    0.4000    1.0000

>> A^10

    1.0000    0.8056    0.5772    0.3126         0
         0    0.0116    0.0168    0.0145         0
         0    0.0135    0.0231    0.0168         0
         0    0.0093    0.0135    0.0116         0
         0    0.1601    0.3694    0.6445    1.0000
```

```
>> X0 = [0;0;1;0;0]

     0
     0
     1
     0
     0

>> A^10 * X0

    0.5772     A wins after 10 games
    0.0168
    0.0231
    0.0135
    0.3694     B wins after 10 games

>>
```

2) Determine the z-transform for the probability that A wins the match after k games

- From the z transforms, determine the explicit function for p(A) wins after game k.

```
>> C = [1,0,0,0,0]

C =     1     0     0     0     0

>> G = ss(A, X0, C, 0, 1);
>> tf(G)

              0.25 z - 0.025
------------------------------------------
z^4 - 1.3 z^3 - 0.07 z^2 + 0.409 z - 0.039

Sampling time (seconds): 1

>> zpk(G)

           0.25 (z-0.1)
-----------------------------------
(z-1) (z-0.7325) (z+0.5325) (z-0.1)

Sampling time (seconds): 1
```

meaning

$$Y = \left( \frac{0.25(z-0.1)}{(z-1)(z-0.7325)(z+0.5325)(z-0.1)} \right) z$$

Using partial fractions

$$Y = \left( \left( \frac{0.60984}{z-1} \right) + \left( \frac{-0.7388}{z-0.7325} \right) + \left( \frac{-0.66553}{z+0.5325} \right) + \left( \frac{0}{z-0.1} \right) \right) z$$

Converting back to time

$$y(k) = \left( 0.60984 - 0.7388(0.7325)^k - 0.66553(-0.5325)^k \right) u(k)$$

3) Two players are playing a game of tennis. To win a game, a player must win 4 points *and* be up by 2 points.

- If player A reaches 4 points and player B has less than 3 points, the game is over and player A wins.
- If player A reaches 4 points and player B has 3 points, then the game reverts to 'win by 2' rules. Both players keep playing until one of them is up by 2 games.

Supppose:

- Player A has a 70% chance of winning any given point
- Player B has a 30% chance of winning any given point.

What is the probabilty that player A wins the game (first to 4 games, win by 2)?

- Note: This is a combination of a binomial distribution (A has 4 points while B has 0, 1, or 2 points) along with a Markov chain (A and B both have 3 points, at which point it becomes a win-by-2 series)

A wins in 4 games: 4-0

$$\text{p=}\binom{4}{4}(0.7)^4(0.3)^0 = 0.2401$$

A wins in 5 games: 4-1

- a wins game 5
- a wins 3 of 4 previous games

$$p = (0.7)\binom{4}{3}(0.7)^3(0.3)^1 = 0.28812$$

A wins in 6 games: 4-2

- A wins game 6
- A wins 3 of 5 previous games

$$p = (0.7)\binom{5}{3}(0.7)^3(0.3)^2 = 0.21609$$

A wins after being 3-3 ( and you enter a Markov chain).

The probability of being 3-3 is

$$p = \binom{6}{3}(0.7)^3(0.3)^3 = 0.18522$$

The probability of A winning once you at 3-3 is a Markov chain like problem #1

$$X(k+1) = \begin{bmatrix} 1 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix} X$$

Inputting this into Matlab and taking A^100 results in

```
>> A = [1,0,0,0,0 ; 0.7,0,0.3,0,0 ; 0,0.7,0,0.3,0 ; 0,0,0.7,0,0.3;0,0,0,0,1]'

A =

    1.0000    0.7000         0         0         0
         0         0    0.7000         0         0
         0    0.3000         0    0.7000         0
         0         0    0.3000         0         0
         0         0         0    0.3000    1.0000

>> A^100

ans =

    1.0000    0.9534    0.8448    0.5914         0
         0    0.0000         0    0.0000         0
         0         0    0.0000         0         0
         0    0.0000         0    0.0000         0
         0    0.0466    0.1552    0.4086    1.0000
```

A has an 84.48% chance of winning given that the game is at 3-3 (love).

The total probability is then

p(A) = p(4-0) + p(4-1) + p(4-2) + p(A wins | 3-3) * p(3-3)

p(A) = 0.2401 + 0.2881 + 0.2161 + (0.8448)(0.1852)

**p(A wins) = 0.9008**

Or use a Monte Carlo simulation

```
% Tennis
p = 0.7;
Awins = 0;

for n=1:1e5

   A = 0;
   B = 0;
   Winner = 0;
   games = 0;
   while( Winner == 0)
       games = games + 1;
       if(rand < p)
           A = A + 1;
       else
           B = B + 1;
       end
       if( (A>=4) * (A >= B+2))
           Winner = 1;
           Awins = Awins + 1;
       end
       if( (B>=4) * (B >= A+2))
           Winner = 2;
       end
   end
  % disp([Winner, games])
end

Awins / 1e5


ans =    0.9003
ans =    0.9015
ans =    0.9003

Elapsed time is 4.143197 seconds.
```

>>

A wins about 90.0% of the matches