ECE 341 - Homework #3

Dice Games and z-Transform

Yahtzee (5 dice)

In the game of Yahtzee, you roll five dice.

- You can then keep whichever dice you like and re-roll the rest.
- You can then do this a second time.

Whatever the results are after three rolls scores points. A Yahtzee is when you roll 5-of-a-kind.

1) Compute the odds of rolling five-of-a-kind when rolling 5 dice one time

dice = xxxxx

(6 values choose 1 for x)(5 spots for x, choose 5)

$$m = \binom{6}{1}\binom{5}{5} = 6$$

$$p = \left(\frac{6}{7776}\right) = 0.000772$$

or

$$p = \binom{6}{1}\left(\frac{1}{6}\right)^5 = 0.000772$$

2) Compute the odds of rolling four-of-a-kind when rolling 5 dice one time

dice = xxxx y

(six values choose 1 for x)(5 spots for x, choose 4)(5 remaining values, choose 1 for y)

$$m = \binom{6}{1}\binom{5}{4}\binom{5}{1} = 150$$

$$p = \left(\frac{150}{7776}\right) = 0.0193$$

3) (Conditional Probability): Compute the odds of getting a Yahtzz (5-of-a-kind) by

- Rolling 4-of-a-kind, then
- Rolling one die and getting a Yahtzee on the next roll, or
- Not getting a Yahtzee on the second roll but getting in on the 3rd roll

A: rolling 5 of a kind

$$p(Y|A) = 1$$

$$p(A) = \left(\frac{6}{7776}\right)$$

B: Rolling 4 of a kind then getting Yahtzee on your next roll

Getting Yahtzee after rolling 4 of a kind

$$p(Y|B) = \left(\frac{1}{6}\right)$$

$$p(B) = \left(\frac{150}{7776}\right)$$

C: Rolling 4 of a kind, not getting Yahtzee on your 2nd roll, but getting it on your 3rd roll

$$p(Y|C) = \left(\frac{1}{6}\right)$$

$$p(C) = \left(\frac{150}{7776}\right)\left(\frac{5}{6}\right)$$

The total odds then are

$$p = p(Y|A)p(A) + p(Y|B)p(B) + p(Y|C)p(C)$$

$$p = (1)\left(\frac{6}{7776}\right) + \left(\frac{1}{6}\right)\left(\frac{150}{7776}\right) + \left(\frac{1}{6}\right)\left(\frac{150}{7776}\right)\left(\frac{5}{6}\right)$$

$$p = 0.006666$$

150 : 1 odds against

4) (Yahtzee program): Write a Matlab program to play Yahtzee.

- Assume you always go for a Yahtzee
- Up to two draw phases (three rolls total)
- Keep the largest pair (2 of a kind, 3-of a kind) and reroll the rest of the dice

```
% ECE 341 Random Proecsses
% Yahtzee Odds
% Monte-Carlo Simulation

tic
disp('------------------');
disp('Yahtzee odds using Monte-Carlo')

Rolls = 3;
Pair5 = 0;

disp('----');
for games = 1:1e5

   Dice = ceil( 6*rand(1,5) );
   Dice = sort(Dice);

   % disp(Dice);

   for turns = 2:Rolls

      % check for pairs

      N = zeros(1,5);
      for i=1:5
         for j=1:5
            if(Dice(i) == Dice(j))
                N(i) = N(i) + 1;
            end
         end
      end
      [N,b] = sort(N, 'descend');
      Dice = Dice(b);

      if (N(1) == 4) Dice(5) = ceil( 6*rand(1,1) ); end
      if (N(1) == 3) Dice(4:5) = ceil( 6*rand(1,2) ); end
      if (N(1) == 2) Dice(3:5) = ceil( 6*rand(1,3) ); end
      if (N(1) == 1) Dice(2:5) = ceil( 6*rand(1,4) ); end
      % disp(Dice);

   end

   N = zeros(1,6);
   for i=1:6
       N(i) = sum(Dice == i);
   end
   [N,b] = sort(N, 'descend');

   if (N(1) == 5) Pair5 = Pair5 + 1; end

end

disp(' Rolls, Yahtzees');
disp([Rolls, Pair5]);

toc
```

5) With your program, do a Monte Carlo simlation for 1 million rolls of the dice and determine the

- Odds of getting a Yahtzee on one roll, and
- Odds of getting a Yahtzee after 3 rolls.

Sanity Check: With one roll (no re-rolls), I got Yahtzees { 92, 90, 74, 83} times in 100,000 rolls. Taking the average

$$p = \left(\frac{84.75}{100,000}\right) = 0.0008475$$

calcualted odds are

$$p = \left(\frac{6}{7776}\right) = 0.000772$$

which is resonably close...

With 100,000 games, the number of Yahtzees is {1225, 1262, 1264, 1242}.

```
Yahtzee odds using Monte-Carlo
----
 Rolls, Yahtzees
           2        1225

Elapsed time is 9.586119 seconds.
```

Taking the average the odds are roughly

$$p = \left(\frac{1248}{100,000}\right) = 0.01248 \text{ or } 80.1 : 1 \text{ against}$$

calcualted odds were

$$p = 0.00666$$

but this only included two ways to get to Yahtzee so it should be a little lower...

6) repeat of problem #2

z-Transforms (over)

Find the inverse z-transform

7) $$X = \left(\frac{0.01}{(z-0.95)(z-0.9)(z-0.85)} \right)$$

multiply by z (planning ahead)

$$zX = \left(\frac{0.01}{(z-0.95)(z-0.9)(z-0.85)} \right) z$$

take the partial fraction expansion

$$zX = \left(\left(\frac{2}{z-0.95} \right) + \left(\frac{-4}{z-0.9} \right) + \left(\frac{2}{z-0.85} \right) \right) z$$

$$zX = \left(\left(\frac{2z}{z-0.95} \right) + \left(\frac{-4z}{z-0.9} \right) + \left(\frac{2z}{z-0.85} \right) \right)$$

Take the inverse z-transform

$$zx(k) = \left(2(0.95)^k - 4(0.9)^k + 2(0.85)^k \right) u(k)$$

Divide by z (delay by one sample)

$$x(k) = \left(2(0.95)^{k-1} - 4(0.9)^{k-1} + 2(0.85)^{k-1} \right) u(k-1)$$

You can also write this as

$$x(k) = \left(2.1053(0.95)^k - 4.444(0.9)^k + 2.3529(0.85)^k \right) u(k-1)$$

8) $$X = \left(\frac{0.1(z+1)(z-1)}{(z-0.9)(z-0.8)(z-0.7)} \right)$$

multiply by z (planning ahead for when we take the inverse z-trnsform)

$$zX = \left(\frac{0.1(z+1)(z-1)}{(z-0.9)(z-0.8)(z-0.7)} \right) z$$

Take the partial fraction expansion

$$zX = \left(\left(\frac{-0.95}{z-0.9} \right) + \left(\frac{3.6}{z-0.8} \right) + \left(\frac{-2.55}{z-0.7} \right) \right) z$$

$$zX = \left(\left(\frac{-0.95z}{z-0.9} \right) + \left(\frac{3.6z}{z-0.8} \right) + \left(\frac{-2.55z}{z-0.7} \right) \right)$$

Take the inverse z-transform

$$zx(k) = \left(-0.95(0.9)^k + 3.6(0.8)^k - 2.5(0.7)^k \right) u(k)$$

Divide by z (delay by one sample)

$$\boxed{x(k) = \left(-0.95(0.9)^{k-1} + 3.6(0.8)^{k-1} - 2.5(0.7)^{k-1} \right) u(k-1)}$$

or equivalently

$$x(k) = \left(-1.0556(0.9)^k + 4.5(0.8)^k - 3.5714(0.7)^k \right) u(k-1)$$

9) A new Tesla Model Y costs $58,990. If you take out a 36-month loan at 2.34% interest, what is your monthly payment? Solve using z-transforms.

The loan value at time k is

$$x(k+1) = \left(1 + \left(\frac{0.0234}{12}\right)\right)x(k) + x(0)\cdot\delta(k) - p\cdot u(k-1)$$

(the loan happens at k = 0, meaning a delta function)

(the monthly payments are a constant for k>0)

where p is your monthly payment starting at k=1 (rather than k=0). Taking the z-transform

$$zX = (1.00195)X + X_0 - p\left(\frac{1}{z-1}\right)$$

Solving for X

$$(z-1.00195)X = X_0 - p\left(\frac{1}{z-1}\right)$$

$$X = \left(\frac{1}{z-1.00195}\right)X_0 - \left(\frac{p}{(z-1)(z-1.00195)}\right)$$

Doing partial fractions

$$zX = \left(\frac{z}{z-1.00195}\right)X_0 - \left(\frac{p}{(z-1)(z-1.00195)}\right)z$$

$$zX = \left(\frac{z}{z-1.00195}\right)X_0 - \left(\left(\frac{512.82p}{z-1.00195}\right) - \left(\frac{512.82p}{z-1}\right)\right)z$$

$$zX = \left(\frac{z}{z-1.00195}\right)X_0 - \left(\frac{512.82pz}{z-1.00195}\right) + \left(\frac{512.82pz}{z-1}\right)$$

$$zx(k) = \left((1.00195)^k X_0 - 512.82p\left((1.00195)^k - 1\right)\right)u(k)$$

divide by z (delay one sample)

$$x(k) = \left((1.00195)^{k-1} X_0 - 512.82p\left((1.00195)^{k-1} - 1\right)\right)u(k-1)$$

At k = 36, the load is zero

$$x(36) = 0 = \left((1.00195)^{35} X_0 - 512.82p\left((1.00195)^{35} - 1\right)\right)$$

The monthly payments are then

$$p = \left(\frac{1}{512.82}\right)\left(\frac{1.00195^{35}}{1-1.00195^{35}}\right)X_0 = \$1745.24$$