ECE 341 - Homework #2

Combinatorics & Card Games. Summer 2023

Combinatorics in Bridge

The card game *bridge* uses a 52-card deck. Each person is dealt 13 cards for their hand.

1) How many different hands are possible? (order doesn't matter)

$$M = \binom{52}{13} = 635,013,559,600$$

What is the probability of having 7 cards of one suit in your hand?

N = (4 suits choose 1) (13 cards of that suit choose 7)(39 other cards choose 6)

$$N = \binom{4}{1}\binom{13}{7}\binom{39}{6}$$

$$N = 22,394,644,272$$

The probability is

$$p = \left(\frac{N}{M}\right) = 0.035266$$

There is a 3.52% chance of being dealt 7 cards of one suit (28.3 : 1 odds against)

#2) What is the probability of having two face-cards (Jacks, Queens, Kings, or Aces)?

N = (16 face cards, choose 2) (36 other cards, choose 11)

$$N = \binom{16}{2}\binom{36}{11} = 72,096,635,520$$

$$p = \left(\frac{N}{M}\right) = 0.113536$$

There is an 11.35% chance you'll have only two face cards in your hand

3) Check your answer using Matlab and a Monte-Carlo simulation

With 100,000 bridge hands, my results were:

- 3073 hands with 7-of-a-suit (3.073%)
- 11508 hands with two face-cards (11.508%)

	Calcualed Odds p	Expected 100,000 Hands	Monte-Carlo 100,000 Hands
7-Card Suit	3.5266%	3,526.6	3,627 3,470 35,17
2 Face Cards	11.3536%	11,353.6	11,298 11,473 11,367

This matches my calculations. Code for problem #3

```
% Bridge
tic
High2 = 0;
Suit7 = 0;

for i0 = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:13);
   Value = mod(Hand-1,13) + 1;
   Suit = ceil(Hand/13);

% check for 7 of a suit
   N = zeros(1,4);
   for n=1:4
      N(n) = sum(Suit == n);
      end

   N = sort(N, 'descend');

   if(N(1) == 7)
      Suit7 = Suit7 + 1;
      end

% check for high-cards
   N = sum(Value == 1) + sum(Value > 10);
   if(N == 2)
      High2 = High2 + 1;
      end

   end

clc
disp('    7 of suit    2 face cards')
disp([Suit7, High2])
toc


     7 of suit    2 face cards
        3627       11298

Elapsed time is 8.123007 seconds.
>>
```

In 6-card poker, you're dealt 6 cards

4) Compute the odds of 2-pair in 6-card poker using combinatorics.

note: your answer should match what you founding using enumeration.

hand = xx yy ab

hands possible

$$M = \binom{52}{6} = 20,358,520$$

Using enumeration, M = 20,358,520 (matches homework #1)

hands that are 2-pair (N = 2,532,816 from enumeration)

N = xx yy ab + xx yy zz

xx yy ab

N = (13c2 for x and y)(4c2 for x)(4c2 for y) * (11c2 for ab)(4c1 for a)(4c1 for b)

$$N_1 = \binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot \binom{11}{2}\binom{4}{1}\binom{4}{1}$$

N1 = 2,471,040

xx yy zz

N2 = (13c3 for xyx)(4c2 for x)(4c2 for y)(4c2 for z)

$$N_2 = \binom{13}{3}\binom{4}{2}\binom{4}{2}\binom{4}{2}$$

N2 = 61,776

N1 + N2 = 2,532,816 (this matches with what we got with enumeration)

The probability is then

$$p = \left(\frac{N}{M}\right) = 12.44\%$$

5) Compute the odds being dealt one-pair using combinatorics

again, your answer should match what you founding using enumeration.

hand = xx a b c d

From enumeration, N = 9,884,160

From combinatorics

N = (13 values for x, choose 1)(4 x's choose 2)(12 values not x choose 4)(4 choose 1 for abcd)

$$N = \binom{13}{1}\binom{4}{2}\binom{12}{4}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}$$

$$N = 9,884,160$$

This matches what we got with enumeration.

The probability is

$$p = \left(\frac{N}{M}\right) = 48.55\%$$

6) Determine the odds of a 2-pair and 1-pair using Matlab and a Monte-Carlo simulation and 100,000 hands of 6-card poker

	Enumeration p	Combinatorics p	Monte-Carlo p
2-Pair	12.44%	12.44%	12.323% 12.404% 12.622%
Pair	48.55%	48.55%	48.642% 48.535% 48.497%

All three methods give similar results. Monte-Carlo results have some error but are close

Problem #6 (code)

```
% 6-Card Stud
tic
Pair22 = 0;
Pair2 = 0;

for i0 = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:6);
   Value = mod((Hand-1),13) + 1;
   Suit = ceil(Hand/13);

   N = zeros(1,13);
   for n=1:13
      N(n) = sum(Value == n);
      end

   [N,a] = sort(N, 'descend');

   if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
   if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end

end

clc
disp('    2-Pair      Pair');
disp([Pair22, Pair2]);
toc
```

```
      2-Pair       Pair
       12323       48642
       12404       48535
       12622       48497

Elapsed time is 10.885058 seconds.
```

Conditional Probability in 6-Card Poker

7) Compute the probability of getting 4-of-a-kind if there is a single draw step

- a) If you are dealt 4-of-a-kind, draw no cards — hand = xxxx draw 0
- b) If you are dealt 3-of-a-kind, draw 3 cards — hand = xxx abc discard abc, draw 3
- c) If you are dealt 2-pair or 2-of-a-kind, draw 4 cards — hand = xx abcd discard abcd, draw 4
- d) If you are dealt no pairs, draw 5 cards. — hand = abcdef discard abcde, draw 5

Using conditional probability

$$p(4) = p(4|A)p(A) + p(4|B)p(B) + p(4|c)p(C) + p(4|D)p(D)$$

a) dealt 4-of-a-kind

$$p(A) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{6}} = 0.000720$$

$$p(4|A) = 1$$

$$p(4|A)p(A) = 0.000720$$

b) Dealt 3-of-a-kind (draw 3 and get the needed card to make 4-of-a-kind)

$$p(B) = \left(\frac{\binom{13}{1}\binom{4}{3}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{6}} \right) = 0.035963$$

$$p(4|B) = \left(\frac{\binom{1}{1}\binom{45}{2}}{\binom{46}{3}} \right) = 0.065217$$

$$p(4|B)p(B) = 0.0023454$$

c) dealt pair or 2-pair

p(C) = 0.1244 + 0.4855 = 0.6099 (problem #4)

Draw 4 cards and get the two you need for 4 of a kind

$$p(4|C) = \left(\frac{\binom{2}{2}\binom{44}{2}}{\binom{46}{4}} \right) = 0.0057971$$

$$p(4|C)p(C) = 0.0035356$$

d) Dealt no pairs, keep one card and draw 5 new

$$p(D) = \left(\frac{\binom{13}{6}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{6}} \right) = 0.3452478$$

$$p(4|D) = \left(\frac{\binom{3}{3}\binom{43}{2}}{\binom{46}{5}} \right) = 0.0006587$$

$$p(4|D)p(D) = 0.0002274$$

Adding it all up

$$p(4) = 0.0068287$$

8) Check your answers using a Monte Carlo simulation with 100,000 of 6-card draw poker

The calculated odds are 0.68287% chanec of getting 4 of a kind.

- With 100,000 hands, you should get 682.87 hands that are four of a kind.

	Calculations p	Calculations 100,000 hands	Monte-Carlo 100,000 hands
4 of a kind	0.0068287	682.287	669 658 689 627 701 646

Code:

```
% ECE 341 Homework #2
% 6 card draw, trying for 4-of-a-kind

tic
Pair4 = 0;

for i0 = 1:1e5
   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:6);
   Value = mod(Hand-1,13) + 1;

   N = Value / 100;
   for i=1:6
       for j=1:6
           if(Value(i) == Value(j)) N(i) = N(i) + 1; end
       end
   end

   [N,b] = sort(N, 'descend');
   N = floor(N);
   Hand = Hand(b);
   Value = mod(Hand,13) + 1;

   if(N(1) == 3)
       Hand(4:6) = Deck(7:9);
   elseif(N(1) == 2)
       Hand(3:6) = Deck(7:10);
   else
       Hand(2:6) = Deck(7:11);
   end

   Value = mod(Hand,13) + 1;

   N = zeros(1,13);
   for n=1:13
       N(n) = sum(Value == n);
   end

   N = sort(N, 'descend');

   if(N(1) == 4) Pair4 = Pair4 + 1; end
   end

disp('6-Card Draw: 4 of a kind ');
disp([Pair4])
toc


6-Card Draw: 4 of a kind
   669
   658
   689

Elapsed time is 12.823158 seconds.
```