ECE 341 - Homework #5

Geometric & Pascal Distributions. Summer 2023

Geometric Distributions

Let A be the number of times you roll an 8-sided die until you get a 1 (p = 1/8)

1) Determine the pdf of A using z-transforms. From this, compute

- The probabilty that A = 10
- The probability that A >= 10

$$A = \left(\frac{1/8}{z-7/8}\right)$$

$$zA = \left(\frac{z \cdot 1/8}{z-7/8}\right)$$

$$za(k) = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^{k} u(k)$$

$$a(k) = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^{k-1} u(k-1)$$

In Matlab

```
>> k=[1:200]';
>> A = 1/8 * (7/8) .^ (k-1);
>> sum(A)

ans =    1.0000            it's a valid pdf:  all probabilities add to one

>> A(10)

ans =    0.0376            p(A = 10)

>> sum(A(10:200))

ans =    0.3007            p(A = 10+)
```

2) Use a Monte-Carlo simulation with 100,000 A's. From your Monte-Carlo simulation, determine

- The probability that A = 10
- The probability that A >= 10

With 100,000 trials

$p(A = 10) = 3803 / 100{,}000$

$= 3.808\%$ vs. 3.76% computed

$p(A >= 10) = 30337 / 100{,}000$

$= 30.337\%$ vs. 30.07% computed

Matlab Code

```
Result = zeros(200,1);
for n=1:1e5
   N = 1;
   while(rand > 1/8)
       N = N + 1;
   end
   Result(N) = Result(N) + 1;
end

Result(10)

ans =        3803

sum(Result(10:200))

ans =       30337
```

Pascal Distribution

Let

- A be the number of times you roll an 8-sided die until you get a 1 (p = 1/8), and
- B be the number of times you roll an 8-sided die until you get a 1 or 2 (p = 1/4).
- X = A + B

3) Determine the pdf of X using z-transforms. From this comptue

- The probability that X = 20
- The probability that X >= 20

$$A = \left(\frac{1/8}{z-7/8}\right)$$

$$B = \left(\frac{1/4}{z-3/4}\right)$$

$$X = AB = \left(\frac{1/8}{z-7/8}\right)\left(\frac{1/4}{z-3/4}\right)$$

$$zX = \left(\frac{1/32}{(z-7/8)(z-3/4)}\right)z$$

$$zX = \left(\left(\frac{0.25}{z-7/8}\right) + \left(\frac{-0.25}{z-3/4}\right)\right)z$$

$$zX = \left(\frac{0.25z}{z-7/8}\right) + \left(\frac{-0.25z}{z-3/4}\right)$$

$$zx(k) = \left(0.25\left(\frac{7}{8}\right)^k - 0.25\left(\frac{3}{4}\right)^k\right)u(k)$$

$$x(k) = \left(\frac{1}{4}\right)\left(\left(\frac{7}{8}\right)^{k-1} - \left(\frac{3}{4}\right)^{k-1}\right)u(k-1)$$

```
>> k = [1:200]';
>> x = 1/4 * ( (7/8).^(k-1) - (3/4).^(k-1) );
>> x(20)

ans =    0.0187              p(x == 20)

>> sum(x(20:200))

ans =    0.1540              p(x >= 20)

>> sum(x)

ans =    1.0000              check: all probabilities add to one

>>
```

4) Determine the pdf of X using convolution. From this, compute

- The probability that X = 20
- The probability that X >= 20

```
>> k = [0:200]';
>> A = 1/8 * (7/8).^(k-1) .* (k>0);
>> B = 1/4 * (3/4).^(k-1) .* (k>0);
>> X = conv(A,B);
>> X = X(1:201);
>> sum(X)

ans =    1.0000

>> X(21)
```

ans =   0.0187   vs. 0.0187 from z-transforms

```
>> sum(X(21:201))
```

ans =   0.1540   vs. 0.1540 from z-transforms

```
>>
```

5) Use a Monte-Carlo simulation with 100,000 X's. From your Monte-Carlo simulation, determine

- The probability that X = 20
- The probability that X >= 20

```
Result = zeros(200,1);
for n=1:1e5
   A = 1;
   while(rand > 1/8)
       A = A + 1;
   end
   B = 1;
   while(rand > 1/4)
       B = B + 1;
   end
   X = A + B;
   Result(X) = Result(X) + 1;
end

Result(20) / 1e5
```

ans = 0.01880   vs. 0.0187 computed

```
sum(Result(20:200)) / 1e5
```

ans =   0.15337   vs. 0.1540 computed

```
>>
```

Pascal Distribution (cont'd)

Let

- A be the number of times you roll a 8-sided die until you roll a 1 (p = 1/8)
- B be the number of times you roll a 8-sided die until you get a 1 or 2 (p = 2/8)
- C be the number of times you roll a 8-sided die until you get a 1, 2, or 3 (p = 3/8)
- Y = A + B + C

6) Determine the pdf of Y using z-transforms. From this comptue

- The probability that Y = 20
- The probability that Y >= 20

$$A = \left(\frac{1/8}{z-7/8}\right)$$
$$B = \left(\frac{1/4}{z-3/4}\right)$$
$$C = \left(\frac{3/8}{z-5/8}\right)$$
$$Y = ABC = \left(\frac{1/8}{z-7/8}\right)\left(\frac{1/4}{z-3/4}\right)\left(\frac{3/8}{z-5/8}\right)$$
$$zY = \left(\frac{3/256}{(z-7/8)(z-3/4)(z-5/8)}\right)z$$
$$zY = \left(\left(\frac{0.375}{z-7/8}\right) + \left(\frac{-0.75}{z-3/4}\right) + \left(\frac{0.375}{z-5/8}\right)\right)z$$
$$zY = \left(\frac{0.375z}{z-7/8}\right) + \left(\frac{-0.75z}{z-3/4}\right) + \left(\frac{0.375z}{z-5/8}\right)$$
$$zy(k) = \left(0.375\left(\tfrac{7}{8}\right)^k - 0.75\left(\tfrac{3}{4}\right)^k + 0.375\left(\tfrac{5}{8}\right)^k\right)u(k)$$
$$y(k) = \left(0.375\left(\tfrac{7}{8}\right)^{k-1} - 0.75\left(\tfrac{3}{4}\right)^{k-1} + 0.375\left(\tfrac{5}{8}\right)^{k-1}\right)u(k-1)$$

```
>> k = [1:200]';
>> y = 0.375 * (7/8).^(k-1) - 0.75*(3/4).^(k-1) + 0.375*(5/8).^(k-1);
>> sum(y)

ans =    1.0000

>> y(20)

ans =    0.0265

>> sum(y(20:200))

ans =    0.2247

>>
```

7) Determine the pdf of Y using convolution. From this, compute

- The probability that Y = 20
- The probability that Y >= 20

```
>> k = [0:200]';
>> A = 1/8 * (7/8).^(k-1);
>> B = 1/4 * (3/4).^(k-1);
>> C = 3/8 * (5/8).^(k-1);
>> AB = conv(A,B);
>> ABC = conv(AB,C);
>> ABC = ABC(1:201);
>> sum(ABC)

ans =    1.0000

>> ABC(21)

ans =    0.0265      vs. 0.0265 computed

>> sum(ABC(21:200))

ans =    0.2247      vs 0.2247 computed
```

8) Use a Monte-Carlo simulation with 100,000 Y's. From your Monte-Carlo simulation, determine

- The probability that Y = 20
- The probability that Y >= 20

With 100,000 rolls,

```
20:      2668        = 2.668%    vs. 2.65% computed

20+      22394       = 22.394%   vs. 22.47% computed
```