ECE 341 - Homework #6

LaPlace Transforms, Continuous Probability Density Functions. Summer 2023

LaPlace Transforms

1) Let X and Y be related by the following transfer function:

$$Y = \left(\frac{10s+20}{(s+3)(s+10)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2 + 13s + 30)Y = (10s + 20)X$$

'sy' means 'the derivative of y(t)'

$$y'' + 13y' + 30y = 10x' + 20x$$

or

$$\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 30y = 10\frac{dx}{dt} + 20x$$

b) Determine y(t) assuming

$$x(t) = 2\cos(4t) + 3\sin(4t)$$

This is a phasor problem

$$s = j4$$

$$X = 2 - j3$$ *real = cosine, -imag = sine*

$$Y = \left(\frac{10s+20}{(s+3)(s+10)}\right)X$$

$$Y = \left(\frac{10s+20}{(s+3)(s+10)}\right)_{s=j4} \cdot (2 - j3)$$

$$Y = 1.1310 - j2.7724$$

meaning

$$y(t) = 1.1310\cos(4t) + 2.7724\sin(4t)$$

c) Determine y(t) assuming x(t) is the unit step function (0 for t<1, 1 for t>0)

$$x(t) = u(t)$$

This is a LaPlace problem (x(t) = 0 for t<0)

$$Y = \left(\frac{10s+20}{(s+3)(s+10)}\right)X$$

$$Y = \left(\frac{10s+20}{(s+3)(s+10)}\right)\left(\frac{1}{s}\right)$$

Do a partial fraction expansion

$$Y = \left(\frac{0.6667}{s}\right) + \left(\frac{0.4762}{s+3}\right) + \left(\frac{-1.1429}{s+10}\right)$$

Take the inverse LaPlace transform

$$\boxed{y(t) = 0.6667 + 0.4762e^{-3t} - 1.1429e^{-10t}} \qquad t>0$$

2) Let X and Y be related by the following transfer funciton

$$Y = \left(\frac{10s+200}{(s+3+j10)(s+3-j10)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2 + 6s + 109)Y = (10s + 200)X$$

$$\boxed{y'' + 6y' + 109y = 10x' + 200x}$$

b) Determine y(t) assuming

$$x(t) = 2\cos(4t) + 3\sin(4t)$$

Use phasors

$$X = 2 - j3$$

$$s = j4$$

$$Y = \left(\frac{10s+200}{(s+3+j10)(s+3-j10)}\right)X$$

$$Y = \left(\frac{10s+200}{(s+3+j10)(s+3-j10)}\right)_{s=j4} \cdot (2 - j3)$$

$$Y = 3.8894 - j6.5951$$

convert back to time

$$y(t) = 3.8894\cos(4t) + 6.5951\sin(4t)$$

c) Determine y(t) assuming x(t) is the unit step function (0 for t<1, 1 for t>0)

$$x(t) = u(t)$$

$$Y = \left(\frac{10s+200}{(s+3+j10)(s+3-j10)}\right)\left(\frac{1}{s}\right)$$

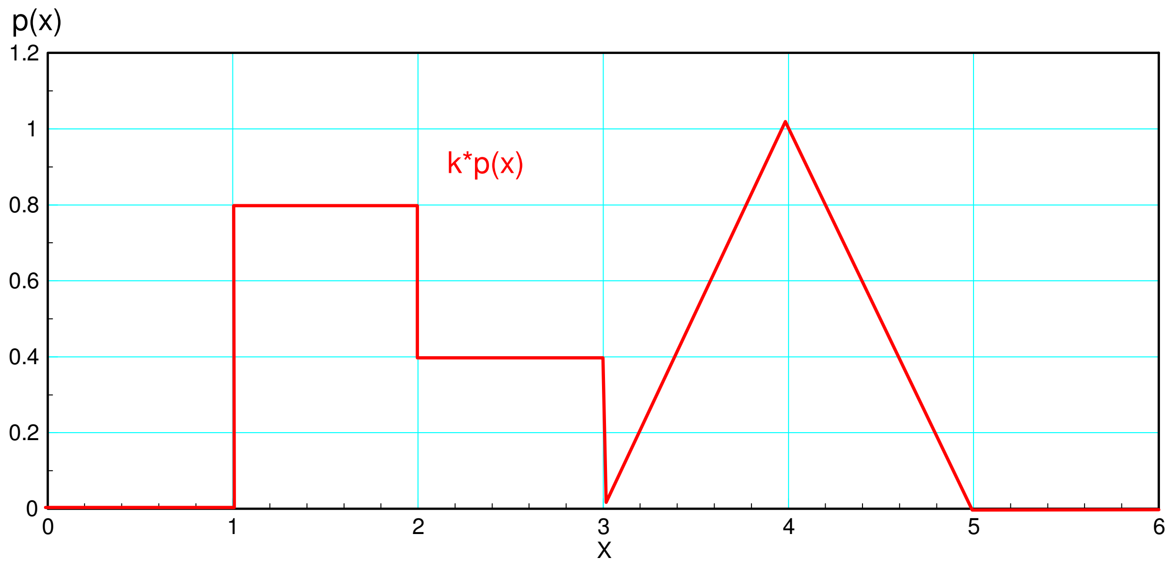
Use partial fractions

$$Y = \left(\frac{1.8349}{s}\right) + \left(\frac{0.9578\angle -163.3^0}{s+3+j10}\right) + \left(\frac{0.9578\angle 163.3^0}{s+3-j10}\right)$$

Convetrt back to time

$$y(t) = 1.8349 + 1.9157e^{-3t}\cos(10t + 163.3^0) \qquad t > 0$$

Continuous Probability Density Functions



3) Determine the scalar so that the above function is a valid pdf (i.e. the total area is 1.000)

Currently, the area is 2.2

Scale the y axis by 1/2.2 = 0.4545

4) Determine the corresponding cdf

$$2.2 \cdot pdf = \begin{cases} 0 & x < 1 \\ 0.8 & 1 < x < 2 \\ 0.4 & 2 < x < 3 \\ x - 3 & 3 < x < 4 \\ 5 - x & 4 < x < 5 \\ 0 & x > 5 \end{cases}$$

Interate

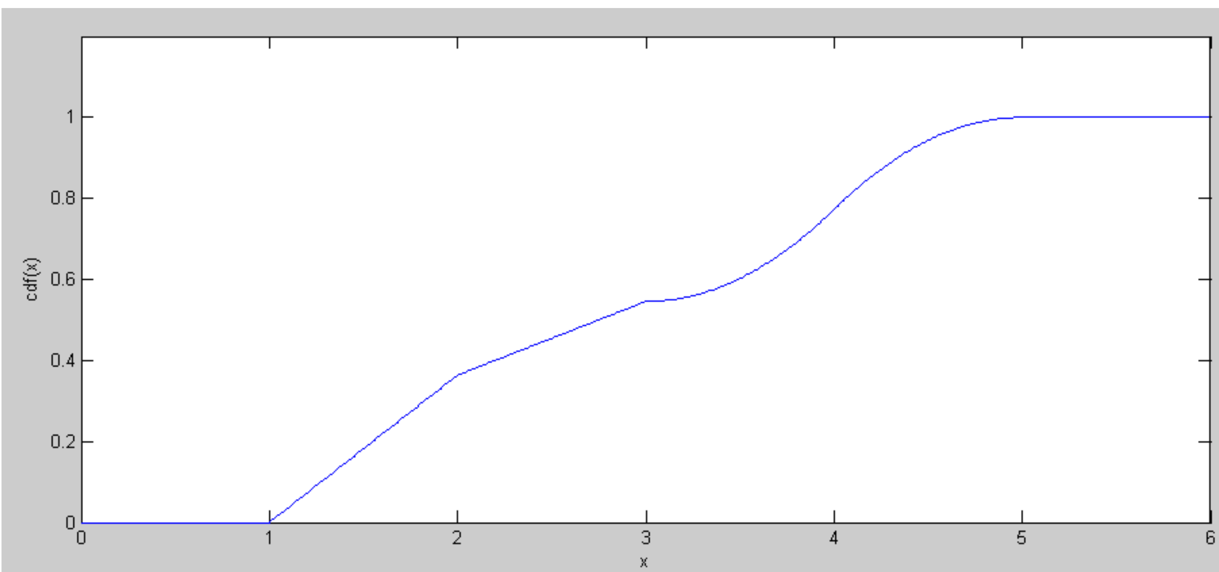
$$2.2 \cdot cdf = \begin{cases} 0 & x < 1 \\ 0.8x - 0.8 & 1 < x < 2 \\ 0.4x & 2 < x < 3 \\ 0.5x^2 - 3x + 5.7 & 3 < x < 4 \\ -0.5x^2 + 5x - 10.3 & 4 < x < 5 \\ 2.2 & x > 5 \end{cases}$$

Checking in Matlab

```
function [y] = cdf(x)

if(x < 1)
    y=0;
elseif(x<2)
    y = 0.8*x-0.8;
elseif(x<3)
    y = 0.4*x;
elseif(x<4)
    y = 0.5*x^2 - 3*x + 5.7;
elseif(x<5)
    y = -0.5*x^2 + 5*x - 10.3;
else
    y=2.2;
end

y = y / 2.2;
end
```

From the command window

```
>> x = [0:0.01:6]';
>> C = 0*x;
>> for i=1:length(x)
    C(i) = cdf(x(i));
    end
>> plot(x,C)
>> ylim([0,1.2])
>> xlabel('x');
>> ylabel('cdf(x)');
```



5) Using Matlab, find 20 random values of x for the above pdf

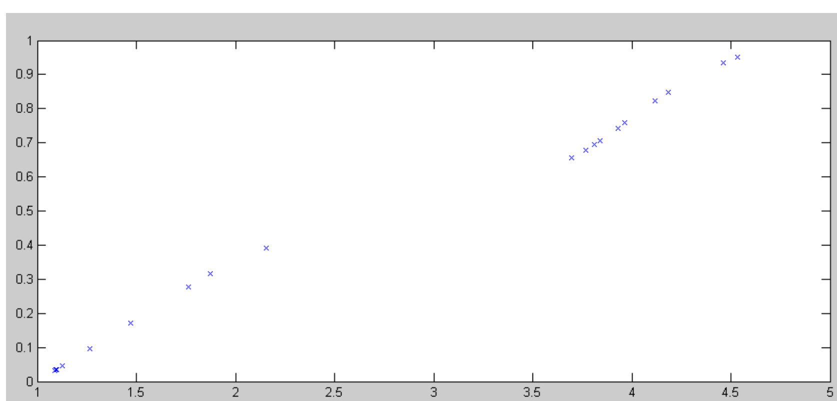
Solve using interval halving:

```
function [x] = Prob5(p)

x1 = 1;
p1 = cdf(x1) - p;
x2 = 5;
p2 = cdf(x2) - p;
for i=1:20
    x3 = (x1+x2)/2;
    p3 = cdf(x3) - p;

    if(p3<0) x1 = x3;
    else     x2 = x3;
    end

%    disp([x1,x2,cdf(x3)])
%    pause(0.5);
end
x = x3;
end
```

Generate 20 random probabilities and corresponding x's

```
>> p = rand(20,1);
>> x = 0*p;
>> for i=1:20
   x(i) = Prob5(p(i));
   end
>> plot(x,p,'x')
>> [p,x]
```

p	x
0.6557	3.6966
0.0357	1.0982
0.8491	4.1852
0.9340	4.4611
0.6787	3.7658
0.7577	3.9665
0.7431	3.9326
0.3922	2.1572
0.6555	3.6958
0.1712	1.4708
0.7060	3.8406
0.0318	1.0875
0.2769	1.7615
0.0462	1.1270
0.0971	1.2671
0.8235	4.1186
0.6948	3.8107
0.3171	1.8720
0.9502	4.5320
0.0344	1.0947

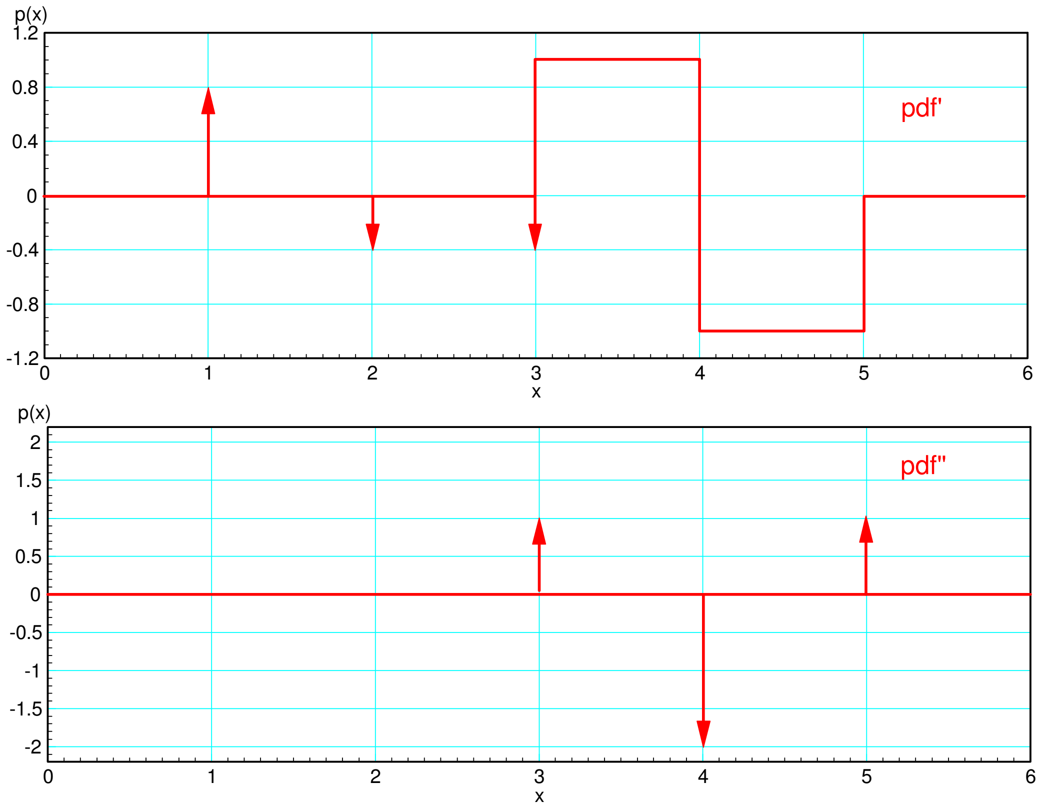



```
>>
```

6) Find the moment generating function for p(x)

a.k.a. Find the LaPlace transform for p(x)

Differntiate until you get delta funtions (or something you recognize)



$$\psi(s) = \left(\frac{0.4545}{s}\right)(0.8e^{-s} - 0.4e^{-2s} - 0.4e^{-3s}) + \left(\frac{0.4545}{s^2}\right)(e^{-3s} - 2s^{-4s} + e^{-5s})$$

The 0.4545 term is include to make this a valid pdf (area is 1.0000)