ECE 341 - Homework #8

Gamma, Poisson, & Normal Distributions. Summer 2023

Gamma Distributions

Let A be an exponential distribution with a mean of 10 seconds

The time until the next customer arrives

Let B be the time until three customers arrive

B has a gamma distribution

1) Determine the pdf for B using LaPlace transforms.

- From your results, determine the pdf at B=20

The moment generating function for the arrival time for one customer is

$$A(s) = \left(\frac{1/10}{s+1/10}\right)$$

The moment generating funciton for the arrival time for three customers is

$$B(s) = \left(\frac{1/10}{s+1/10}\right)^3$$

Using a table of LaPlace transforms

$$t^2 e^{-bt} u(t) \leftrightarrow \left(\frac{2}{(s+b)^3}\right)$$

$$B(s) = \left(\frac{1}{2000}\right)\left(\frac{2}{(s+1/10)^3}\right)$$

$$\boxed{b(t) = \left(\frac{1}{2000}\right) t^2 e^{-t/10} u(t)}$$

$$b(20) = \left(\frac{1}{2000}\right)(20)^2 e^{-2}$$

$$\boxed{b(20) = 0.0271}$$

2) Determine the pdf of B using convolution

- From your results, determine the pdf at B = 20

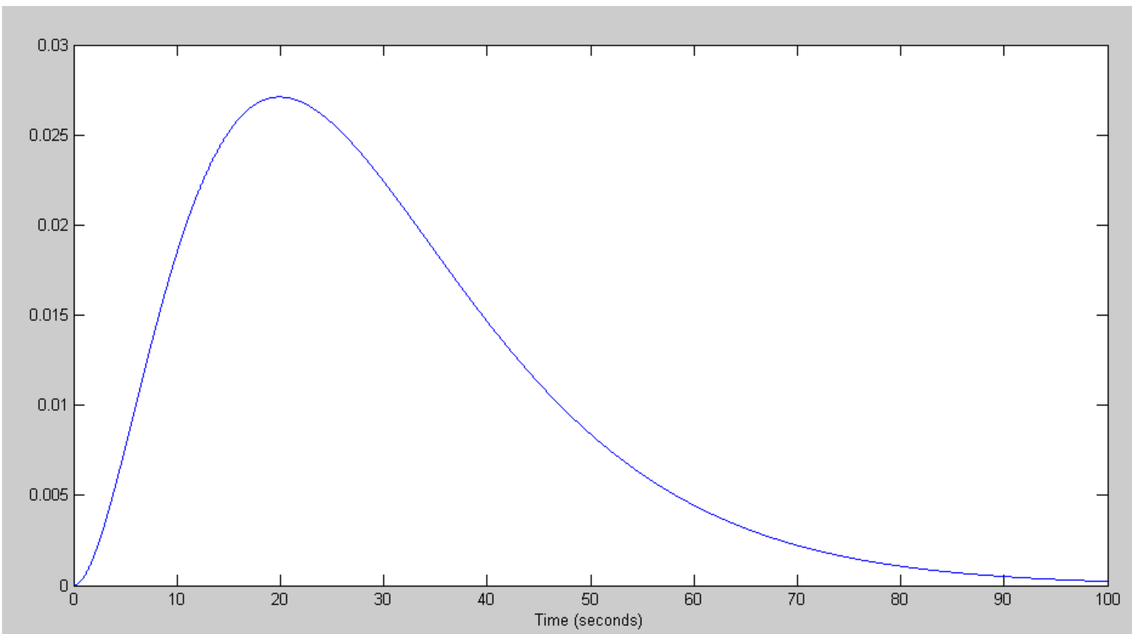
```
>> dt = 0.01;
>> t = [0:dt:100]';
>> A = 0.1 * exp(-t/10);
>> A2 = conv(A,A) * dt;
>> A3 = conv(A2,A) * dt;
>> B = A3(1:length(t));
>> t(2001)

ans =    20

>> B(2001)
```

ans =   0.0271   *matches problem #1*

```
>> plot(t,B)
```



Poisson Distributions

3) Determine the probability that 3 customers will arrive within 20 seconds (0 < t < 20)

Method #1: Poisson pdf:

$$f(k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right)$$

$$\lambda = np = (20)\left(\frac{1}{10}\right) = 2$$

$$f(k=3) = \left(\frac{1}{3!}\right)(2)^3 e^{-2}$$

$$f(k=3) = 18.04\%$$

Method #2: Integrate the pdf for a Gamma distribution.

The pdf for three customers arriving at t seconds is

$$\left(\frac{1/10}{s+1/10}\right)^3$$

The cdf is the integral of the pdf

$$cdf = \left(\frac{1/10}{s+1/10}\right)^3 \cdot \left(\frac{1}{s}\right)$$

Doing a partial fraction expansion

$$cdf = \left(\frac{1}{s}\right) + \left(\frac{A}{(s+1/10)^3}\right) + \left(\frac{B}{(s+1/10)^2}\right) + \left(\frac{C}{s+1/10}\right)$$

Putting over a common denominator and solving for A, B, C

$$\left(s+\frac{1}{10}\right)^3 + As + Bs\left(s+\frac{1}{10}\right) + Cs\left(s+\frac{1}{10}\right)^2 = \left(\frac{1}{1000}\right)$$

$$\left(s^3 + \frac{3}{10}s^2 + \frac{3}{100}s + \frac{1}{1000}\right) + As + B\left(s^2 + \frac{1}{10}s\right) + Cs\left(s^2 + \frac{2}{10}s + \frac{1}{100}\right) = \left(\frac{1}{1000}\right)$$

$$C = -1$$

$$B = \frac{-1}{10}$$

$$A = \frac{-1}{100}$$

$$CDF(s) = \left(\frac{1}{s}\right) + \left(\frac{-0.01}{(s+1/10)^3}\right) + \left(\frac{-0.1}{(s+1/10)^2}\right) + \left(\frac{-1}{s+1/10}\right)$$

$$cdf(t) = \left(1 + \left(\frac{-0.01}{2}\right)t^2 e^{-t/10} - 0.1te^{-t/10} - e^{-t/10}\right)u(t)$$

$$cdf(t) = \left(1 - \left(\left(\frac{0.01}{2}\right)t^2 + 0.1t + 1\right)e^{-t/10}\right)u(t)$$

$$cdf(20) = 1 - 5e^{-2}$$

$$cdf(20) = 0.3233$$

Using convolution and numerical integration

```
>> dt = 0.01;
>> t = [0:dt:100]';
>> A = 0.1 * exp(-t/10);
>> A2 = conv(A,A) * dt;
>> A3 = conv(A2,A) * dt;
>> B = A3(1:length(t));

>> sum(B(1:2001)) * dt

ans =    0.3244
```

which is close to the exact answer from the moment-generating funciton

cdf(20) = 0.323324

A smaller step size or extending time out to 200 would give a more-exact answer

4) In D&D, you automatically make your saving throw if you roll a 20 on a 20-sided die (p = 5%).

- Using a binomial pdf, determine the probability of making your saving throw four times in 20 rolls
- Using a Poisson approximation, determine the probability of making four saving throws in 20 rolls

Binomial

$$p = \binom{20}{4}\left(\frac{1}{20}\right)^4\left(\frac{19}{20}\right)^{16}$$

$$p = 0.013328$$

Poisson

$$\lambda = np = 20 \cdot \left(\frac{1}{20}\right) = 1$$

$$f(x) = \frac{1}{x!} \cdot \lambda^x e^{-\lambda}$$

$$f(4) = \frac{1}{4!} \cdot (1)^4 \cdot e^{-1}$$

$$f(4) = 0.015328$$

Normal Distribution

- Let x be a random number from a normal distribution with a mean of 10 and a standard deviation of 6
- Let y be a random number from a normal distribution with a mean of 15 and a standard deviation of 8
- Let z be a random number from a normal distribution with a mean of 20 and a standard deviation of 10

5) Let F = x + y. Determine the probability that F > 40

- a) Using a z-score
- b) Using a Monte-Carlo simulation with 100,000 samples of F

z-Score: Normal + Normal = Normal

- means add
- variances add

$$\mu_f = \mu_x + \mu_y$$

$$\mu_f = 10 + 15 = 25$$

$$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$$

$$\sigma_f^2 = 6^2 + 8^2 = 100$$

$$\sigma_f = 10$$

The z-score is the distance to the mean in terms of stanard deviations

$$z = \left(\frac{40-25}{10}\right) = 1.50$$

From StatTrek, a z-score of 1.50 corresponds to a tail with an area of 0.06681

There is a 6.681% chance that F > 40

Monte-Carlo Simulation

```
W = 0;
for n=1:1e6
   A = randn*6 + 10;
   B = randn*8 + 15;
   F = A + B;

   if(F > 40)
       W = W + 1;
   end
end
W / 1e6

ans =    0.0670       about the same answer
```

6) Let G = x + y + z. Determine the probability that G > 60

- a) Using a z-score
- b) Using a Monte-Carlo simulation with 100,000 samples of G

$$\mu_g = \mu_x + \mu_y + \mu_z$$
$$\mu_g = 10 + 15 + 20$$
$$\mu_g = 45$$

$$\sigma_g^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$$
$$\sigma_g^2 = 6^2 + 8^2 + 10^2$$
$$\sigma_g^2 = 200$$
$$\sigma_g = 14.1421$$

The z-score for 60.5 is

$$z = \left(\frac{60-45}{14.1421}\right) = 1.0607$$

From a normal distribution table, this corresponds to p = 0.8554 (1 - p = 0.1446)

There is a 14.46% chance of the sum being more than 60

Monte-Carlo with 1 million rolls

```
W = 0;
for n=1:1e6
   A = randn*6 + 10;
   B = randn*8 + 15;
   C = randn*10 + 20;
   G = A + B + C;

   if(G > 60)
       W = W + 1;
   end
end
W / 1e6

ans =     0.1444
```

There is a 14.44% chance of rolling more than 60