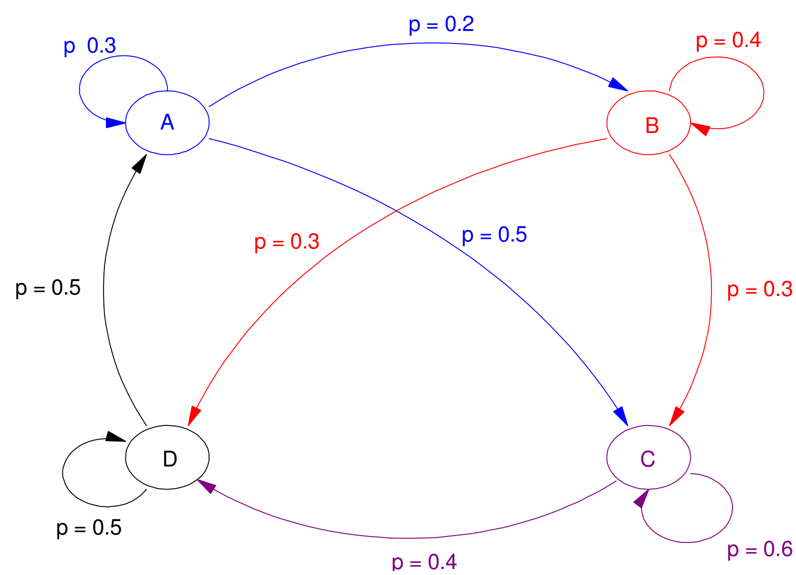
ECE 341 - Homework #11

Markov Chains. Summer 2023

Markov Chains

Four people are playing a game of hot potato. Each second, a player can keep the potato or pass it to another player. The probability of each decision are as follows:



1) Assume player A starts with the potato. Determine the probability that each player has the potato after 10 tosses using matrix multiplication.

$$\begin{bmatrix} A(k+1) \\ B(k+1) \\ C(k+1) \\ D(k+1) \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0 & 0.5 \\ 0.2 & 0.4 & 0 & 0 \\ 0.5 & 0.3 & 0.6 & 0 \\ 0 & 0.3 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} A(k) \\ B(k) \\ C(k) \\ D(k) \end{bmatrix}$$

```
>> M = [0.3,0,0,0.5 ; 0.2,0.4,0,0 ; 0.5,0.3,0.6,0 ; 0,0.3,0.4,0.5]

    0.3000         0         0    0.5000
    0.2000    0.4000         0         0
    0.5000    0.3000    0.6000         0
         0    0.3000    0.4000    0.5000

>> X0 = [1;0;0;0]

     1
     0
     0
     0

>> M^10 * X0
```

A(10) 0.2363
B(10) 0.0785
C(10) 0.3540
D(10) 0.3311

2) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using z-transforms.

- What is the probability that player A has the potato after infinite tosses?

```
>> A = M

    0.3000         0         0    0.5000
    0.2000    0.4000         0         0
    0.5000    0.3000    0.6000         0
         0    0.3000    0.4000    0.5000

>> X0 = [1;0;0;0]

     1
     0
     0
     0

>> C = [1,0,0,0]

     1     0     0     0

>> D = 0;
>> G = ss(A,X0,C,D,1);

>> zpk(G)

         (z-0.5) (z-0.6) (z-0.4)
-----------------------------------------
(z-1) (z-0.3532) (z^2 - 0.4468z + 0.2322)

Sampling time (seconds): 1
```

Multiply by z to get the z-transform for A(z)

$$A(z) = \left(\frac{(z-0.4)(z-0.5)(z-0.6)z}{(z-1)(z-0.3532)(z^2-0.4468z+0.2322)}\right)$$

Do a partial fraction expansion

$$A(z) = \left(\left(\frac{0.2362}{z-1}\right) + \left(\frac{0.0132}{z-0.3532}\right) + \left(\frac{0.3963\angle-18.71^0}{z-0.4819\angle 62.376^0}\right) + \left(\frac{0.3969\angle+18.71^0}{z-0.4819\angle-62.376^0}\right)\right)z$$

Convert back to k

$$a(k) = 0.2362 + 0.0132(0.3532)^k + 0.7925(0.4819)^k \cos(62.376k^0 + 18.71^0)$$

3) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using eigenvalues and eigenvectors.

ate-transistion matrix:

```
>> A

    0.3000         0         0    0.5000
    0.2000    0.4000         0         0
    0.5000    0.3000    0.6000         0
         0    0.3000    0.4000    0.5000
```

Find the eigenvalues & eigenvectors. The one we care about is the eigenvector associated with the eigenvalue at z = +1 (show in red)

```
>> [M,V] = eig(A)

M =

  -0.4335            -0.6190             -0.6190            -0.1849
  -0.1445             0.1024 + 0.2476i    0.1024 - 0.2476i   0.7895
  -0.6503             0.4218 + 0.2809i    0.4218 - 0.2809i  -0.5849
  -0.6069             0.0948 - 0.5285i    0.0948 + 0.5285i  -0.0197

eigenvalues
   1.0000             0.2234 + 0.4269i    0.2234 - 0.4269i   0.3532
```

A starts with the potato

```
>> X0 = [1;0;0;0];
```

which gives you the following initial conditions

```
>> IC = inv(M)*X0

  -0.5449 - 0.0000i
  -0.6063 - 0.2054i
  -0.6063 + 0.2054i
  -0.0713 + 0.0000i
```

so X(k) is

$$x(k) = -0.5449 \begin{bmatrix} -0.4335 \\ -0.1445 \\ -0.6503 \\ -0.6069 \end{bmatrix} (1)^k + (0.2234 + j0.4269) \begin{bmatrix} -0.6190 \\ -0.1024 + j0.2476 \\ 0.4218 + j0.2809 \\ 0.0948 - j0.5285 \end{bmatrix} (0.2234 + j0.4269)^k$$

$$+(-0.6063 + j0.2054) \begin{bmatrix} -0.6190 \\ 0.1024 - j0.2476 \\ 0.4218 - j0.2809 \\ 0.0948 + j0.5285 \end{bmatrix} (0.2234 - j0.4269)^k$$

$$+0.3432 \begin{bmatrix} -0.1849 \\ 0.7895 \\ -0.5849 \\ -0.1987 \end{bmatrix} (0.3532)^k$$

If you multiply the eigenvector by the scalar out front you get an equivalent answer

```
>> a = IC(1);
>> b = IC(2);
>> c = IC(3);
>> d = IC(4);

>> [a*M(:,1), b*M(:,2), c*M(:,3), d*M(:,4)]

ans =

   0.2362 + 0.0000i   0.3753 + 0.1272i   0.3753 - 0.1272i   0.0132 - 0.0000i
   0.0787 + 0.0000i  -0.0112 - 0.1712i  -0.0112 + 0.1712i  -0.0563 + 0.0000i
   0.3543 + 0.0000i  -0.1980 - 0.2570i  -0.1980 + 0.2570i   0.0417 - 0.0000i
   0.3307 + 0.0000i  -0.1661 + 0.3010i  -0.1661 - 0.3010i   0.0014 - 0.0000i
```

eigenvalues

```
>>    1.0000             0.2234 + 0.4269i   0.2234 - 0.4269i   0.3532
```

meaning

$$X(k) = \begin{bmatrix} 0.2362 \\ 0.0787 \\ 0.3543 \\ 0.3307 \end{bmatrix}(1)^k + \begin{bmatrix} 0.3753 + j0.1272 \\ -0.0112 + j0.1712 \\ -0.1980 - j0.2570 \\ -0.1661 + j0.3010 \end{bmatrix}(0.2234 + j0.4269)^k$$

$$+ \begin{bmatrix} 0.3753 - j0.1272 \\ -0.0112 - j0.1712 \\ -0.1980 + j0.2570 \\ -0.1661 - j0.3010 \end{bmatrix}(0.2234 - j0.4269)^k + \begin{bmatrix} 0.0132 \\ -0.0563 \\ 0.0417 \\ 0.0014 \end{bmatrix}(0.3532)^k$$

Markov Chains with Absorbing States

Problem 4 & 5: Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 60% chance of winning
- There is a 15% chance of a tie, and
- Team B has a 25% chance of winning

In order to win the match, a team must be up by 2 games.

4) Determine the probabilty that team A wins the match after k games for k = {0 ... 10} using matrix multiplication.

relationship between the points after one game is:

$$\begin{bmatrix} P2(k+1) \\ P1(k+1) \\ P0(k+1) \\ M1(k+1) \\ M2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.15 & 0.6 & 0 & 0 \\ 0 & 0.25 & 0.15 & 0.6 & 0 \\ 0 & 0 & 0.25 & 0.15 & 0 \\ 0 & 0 & 0 & 0.25 & 1 \end{bmatrix} \begin{bmatrix} P2(k) \\ P1(k) \\ P0(k) \\ M1(k) \\ M2(k) \end{bmatrix}$$

Using Matlab and matrix multiplication:

```
>> A = [1,0.6,0,0,0 ; 0,0.15,0.6,0,0 ; 0,0.25,0.15,0.6,0 ; 0,0,0.25,0.15,0 ;
0,0,0,0.25,1]

    1.0000    0.6000         0         0         0
         0    0.1500    0.6000         0         0
         0    0.2500    0.1500    0.6000         0
         0         0    0.2500    0.1500         0
         0         0         0    0.2500    1.0000

>> X0 = [0;0;1;0;0]

     0
     0
     1
     0
     0

C = [1,0,0,0,0];

for n=0:10
   y = C * A^n * X0;
   disp([n,y]);
   end

     0      0
     1      0
     2    0.3600
     3    0.4680
     4    0.6003
     5    0.6700
     6    0.7276
     7    0.7642
     8    0.7912
     9    0.8094
    10    0.8224
```

5) Determine the z-transform for the probability that A wins the match after k games

- From the z transforms, determine the explicit function for p(A) wins after game k.

```
>> G = ss(A, X0, C, 0, 1);
>> zpk(G)

            0.36 (z-0.15)
------------------------------------
(z-1) (z-0.6977) (z+0.3977) (z-0.15)
```

meaning

$$A(z) = \left(\frac{0.36z(z-0.15)}{(z-1)(z-0.6977)(z+0.3977)(z-0.15)} \right)$$

Do a partial fraction expansion

$$A(z) = \left(\left(\frac{0.8521}{z-1} \right) + \left(\frac{-1.0872}{z-0.6977} \right) + \left(\frac{0.2351}{z-0.3977} \right) + \left(\frac{0}{z-0.15} \right) \right) z$$

Take the inverse z-transform

$$a(k) = \left(0.8521 - 1.0872(0.6977)^k + 0.2351(-0.3977)^k \right) u(k)$$

Comparing the two answers

```
for k=0:10
   y = C * A^k * X0;
   a = 0.8521 - 1.0872*(0.6977)^k + 0.2351*(-0.3977)^k;
   disp([k,y,a]);
   end
```

k	prob 4	prob 5
0	0	0.0000
1.0000	0	0.0001
2.0000	0.3600	0.3601
3.0000	0.4680	0.4681
4.0000	0.6003	0.6004
5.0000	0.6700	0.6700
6.0000	0.7276	0.7276
7.0000	0.7642	0.7642
8.0000	0.7912	0.7912
9.0000	0.8094	0.8094
10.0000	0.8224	0.8224

Problem 6: Two players are playing a game of tennis. To win a game, a player must win 3 points *and* be up by 2 points.

- If player A reaches 3 points and player B has less than 2 points, the game is over and player A wins.
- If player A reaches 2 points and player B has 2 points, then the game reverts to 'win by 2' rules. Both players keep playing until one of them is up by 2 games.

Suppose:

- Player A has a 60% chance of winning any given point
- Player B has a 40% chance of winning any given point.

What is the probabilty that player A wins the game (first to 4 games, win by 2)?

- Note: This is a combination of a binomial distribution (A has 4 points while B has 0, 1, or 2 points) along with a Markov chain (A and B both have 3 points, at which point it becomes a win-by-2 series)

This is a conditional probability. The ways A can win is

A wins 3-0 binomial distribution

A wins 3-1 binomial distribution

A wins a tie-breaker after going 2-2

From a binomial distribution, the odds are:

3-0:

$$p = \binom{3}{3}(0.6)^3(0.4)^0 = 0.2160$$

3-1 = A wins after going 2-1

$$p = 0.6 \cdot \binom{3}{2}(0.6)^2(0.4)^1 = 0.2592$$

2-2

$$p = \binom{4}{2}(0.6)^2(0.4)^2 = 0.3456$$

If you are at 2-2, it's now a Markov chain where you have to win by two. The state-transistion matrix is

$$\begin{bmatrix} P2(k+1) \\ P1(k+1) \\ P0(k+1) \\ M1(k+1) \\ M2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} P2(k) \\ P1(k) \\ P0(k) \\ M1(k) \\ M2(k) \end{bmatrix}$$

The probability that A wins given you start at 2-2 is

```
>> A = [1,0.6,0,0,0 ; 0,0,0.6,0,0 ; 0,0.4,0,0.6,0 ; 0,0,0.4,0,0 ; 0,0,0,0.4,1]

    1.0000    0.6000         0         0         0
         0         0    0.6000         0         0
         0    0.4000         0    0.6000         0
         0         0    0.4000         0         0
         0         0         0    0.4000    1.0000

>> X0 = [0;0;1;0;0]

     0
     0
     1
     0
     0

>> A^100 * X0

    0.6923
         0
    0.0000
         0
    0.3077
```

The total odds are then a conditional probability

$$p(A) = p(A|3-0)p(3-0) + p(A|3-1)p(3-1) + p(A|2-2)p(2-2)$$

$$p(A) = 1 \cdot 0.2160 + 1 \cdot 0.2592 + 0.6923 \cdot 0.3456$$

$$p(A) = 0.7145$$

With this format, player A has a 71.45% chance of winning any given match