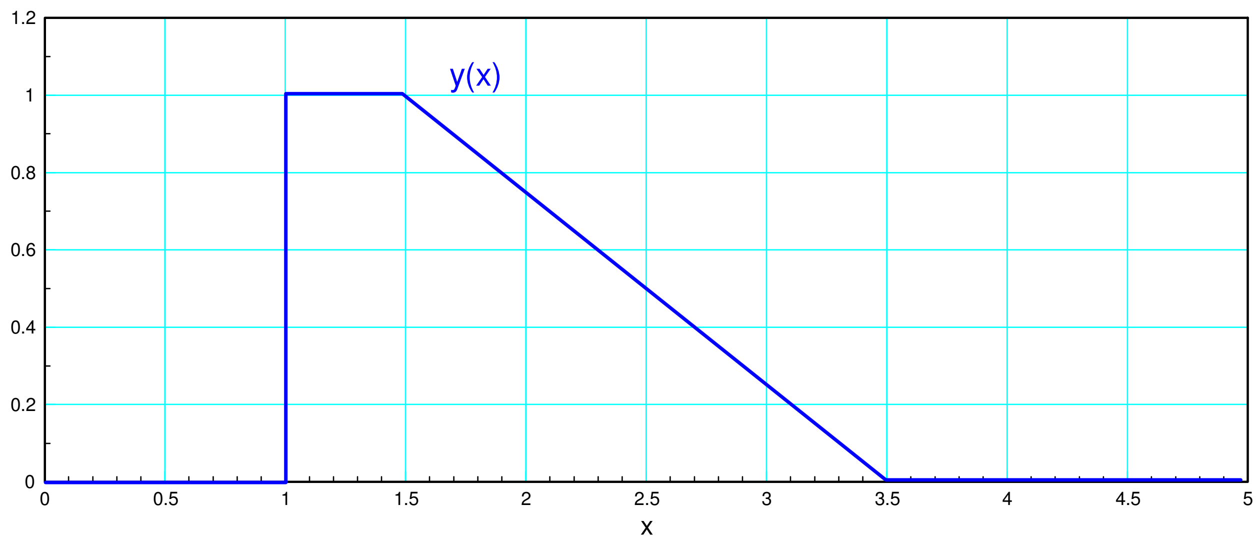
ECE 341 - Test #2

Continuous Probability - Summer 2023

1) Continuous PDF

Let

$$y = \begin{cases} \alpha \cdot 1 & 1 > x > 1.5 \\ \alpha \cdot (1.75 - 0.5x) & 1.5 < x < 3.5 \\ 0 & otherwise \end{cases}$$



a) Determine the scalar, α, so that this is a valid pdf (i.e. the total area = 1.0000)

The area under the curve is 3/2

$$\alpha = 2/3$$

b) Determine the moment generating function (i.e. LaPlace transform)

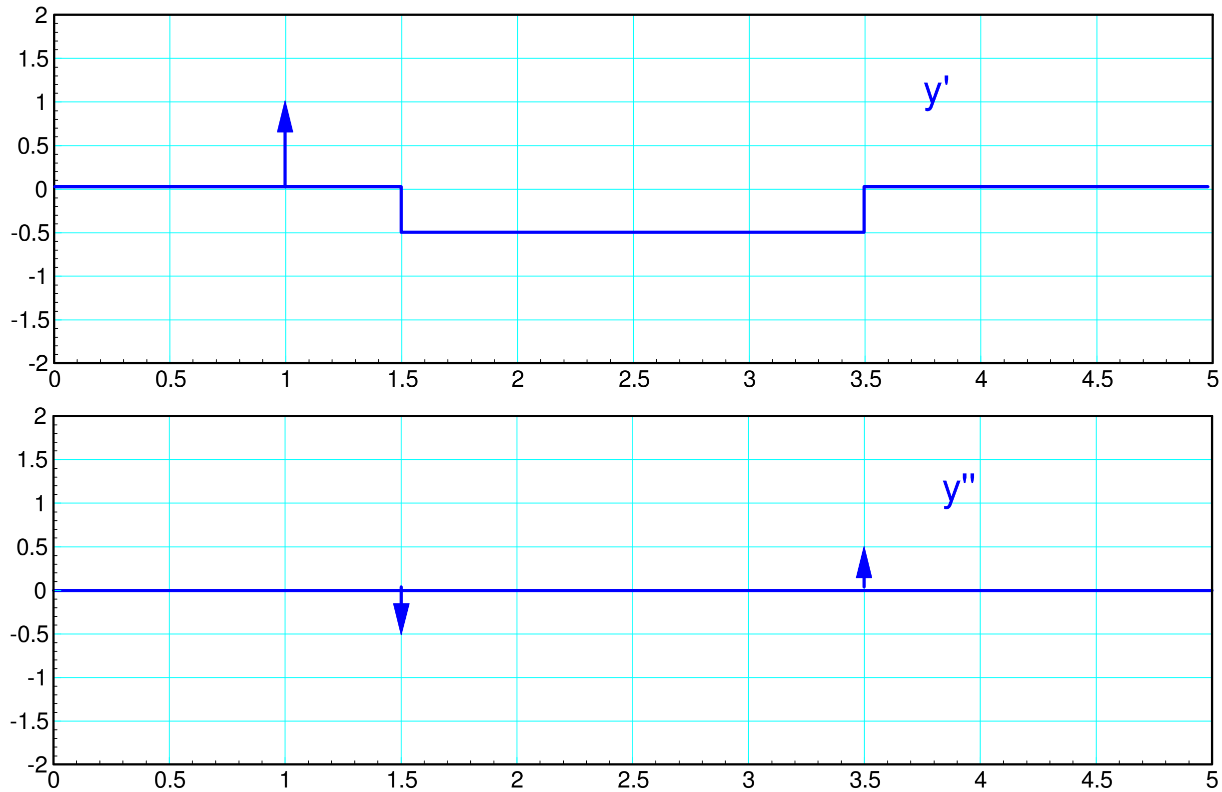
Take the derivatives and note where the delta functions are

y': $sY = 1 \cdot e^{-s}$

y'': $s^2 Y = \left(-\frac{1}{2}e^{-1.5s} + \frac{1}{2}e^{-3.5s}\right)$

Putting it together

$$Y = \left(\frac{2}{3s}\right)(e^{-s}) + \left(\frac{2}{3s^2}\right)\left(-\frac{1}{2}e^{-1.5s} + \frac{1}{2}e^{-3.5s}\right)$$



2) Uniform Distribuitions

Let A, B, and C be continuous uniform distributions

- A = uniform over the interval of (0, 2)
- B = uniform over the interval of (1, 4),
- X = A + B

Use moment generating functions to determine the pdf for X (i.e. LaPlace Transforms)

Start with the LaPlace transform of A and B

$$A(s) = \left(\frac{1}{2s}\right)(1 - e^{-2s})$$

$$B(s) = \left(\frac{1}{3s}\right)(e^{-s} - e^{-4s})$$

In the LaPlace domain, A(s) and B(s) multiply to produce X(s)

$$X(s) = \left(\frac{1}{2s}\right)(1 - e^{-2s}) \cdot \left(\frac{1}{3s}\right)(e^{-s} - e^{-4s})$$

Doing some algebra

$$X(s) = \left(\frac{1}{6s^2}\right)(e^{-s} - e^{-3s} - e^{-4s} + e^{-6s})$$

using LaPlace transforms:

$$\left(\frac{1}{s^2}\right) \rightarrow tu(t)$$

$$e^{-Ts}F(s) \rightarrow f(t-T)$$

x(t) is then

$$x(t) = \left(\frac{1}{6}\right)((t-1)u(t-1) - (t-3)u(t-3) - (t-4)u(t-4) + (t-6)u(t-6))$$

3) Gamma CDF

Let A, B be continuous exponential distributions:

- A has a mean of 3 seconds $a(t) = \frac{1}{3}e^{-t/3}u(t)$ $A(s) = \left(\frac{1/3}{s+1/3}\right)$
- B has a mean of 4 seconds $b(t) = \frac{1}{4}e^{-t/4}u(t)$ $B(s) = \left(\frac{1/4}{s+1/4}\right)$

Determine the equation for the cdf (i.e. the integal of the pdf) for Y = A + B using moment generating functions (i.e. LaPlace transforms)

The cdf is

$$Y(s) = \left(\frac{1}{s}\right) \cdot A(s) \cdot B(s)$$

$$Y(s) = \left(\frac{1}{s}\right)\left(\frac{1/3}{s+1/3}\right)\left(\frac{1/4}{s+1/4}\right)$$

Doing a partial fraction expansion

$$Y(s) = \left(\frac{1}{s}\right) + \left(\frac{3}{s+1/3}\right) + \left(\frac{-4}{s+1/4}\right)$$

Take the inverse LaPlace transform to get the cdf

$$y(t) = (1 + 3e^{-t/3} - 4e^{-t/4})u(t)$$

4) Central Limit Theorem

The Dungeons and Dragons spell *Insect Plague* does 4-40 damage (the sum of four 10-sided dice). Use a normal approximation to determine the probability that the total damage is less than 9.5.

Note: for a single 10-sided die

- mean = 5.5
- variance = 8.25

mean of 4d10	standard deviatio of 4d10	z-score for sum = 9.5	p(sum < 9.5)
22.00	**5.7456**	**2.17597**	**1.478%**

The mean and variance scale with the number of dice

$$\mu = 4 \cdot 5.5 = 22$$

$$\sigma^2 = 4 \cdot 8.25 = 33$$

$$\sigma = \sqrt{33} = 5.7456$$

The z-score for 9.5 damage is the distance to the mean in terms of standard deviations

$$z = \left(\frac{22-9.5}{5.7456}\right) = 2.17558$$

Use a normal table to convert this to a probability

$$p = 0.01479$$

There is a 1.478% chance of doing less than 9.5 damage with an *Insect Plague* spell

Checking with a Monte-Carlo simulation

```
N = 0;
for n=1:1e6
    D = sum( ceil( 10*rand(1,4) ) );
    if(D < 9.5) N = N + 1; end
end
N/1e6

ans =    0.012634
```

5) Testing with Normal pdf

Let

- x have a uniform distribution over the range of (1, 10)
- A be the sum of three x's (range = 3..30)
- B be the sum of four x's (range = 4..40)

Use a normal approximation to determine the probability that A > B

Note: The mean and variance for x (a uniform distribution over the range of (1,10)) is

- mean(x) = 5.5
- variance(x) = 6.75

A:

$$\mu_a = 3 \cdot 5.5 = 16.5$$

$$\sigma_a^2 = 3 \cdot 6.75 = 20.25$$

B:

$$\mu_b = 4 \cdot 5.5 = 22.0$$

$$\sigma_b^2 = 4 \cdot 6.75 = 27.00$$

W = A - B

$$\mu_w = \mu_a - \mu_b = -5.5$$

$$\sigma_w^2 = \sigma_a^2 + \sigma_b^2 = 47.25$$

$$\sigma_w = \sqrt{47.25} = 6.8738$$

The z-score for A > B is

$$z = \left(\frac{\mu_w - 0}{\sigma_w}\right) = \left(\frac{-5.5}{6.8738}\right)$$

$$z = -0.80013$$

Use a normal distribution table to convert this to a probability

p = 0.21182

Player A has a 21.182% chance of beating player B

Running a Monte-Carlo simulation to check:

```
N = 0;
for n=1:1e6
    A = sum( 1 + 9*rand(1,3) );
    B = sum( 1 + 9*rand(1,4) );
    if(A > B) N = N + 1; end
end
N / 1e6

ans =    0.2158
```

Monte-Carlo gives a 21.58% chance of A winning