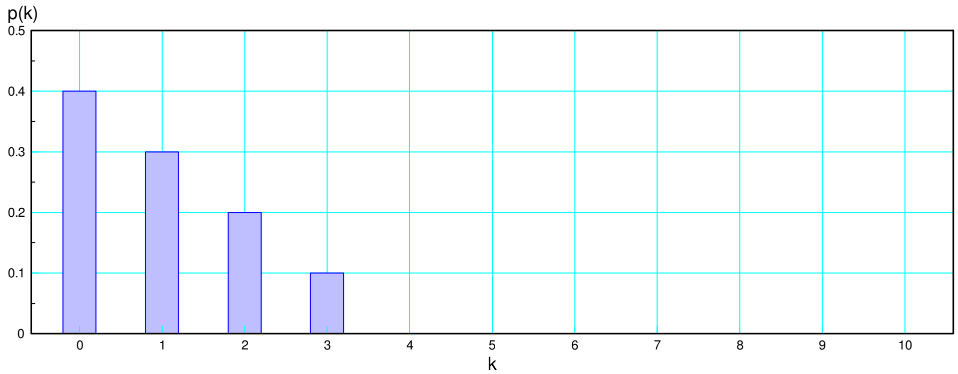
z-Transforms



Discrete pdf with a z-transform of $\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$

z-Transform:

z-transforms are similar to LaPlace transforms - only they apply to discrete-time systems (or discrete probability density functions). The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Hence, the above figure which has the pdf of

$$pdf = 0.4\ \delta(k) + 0.3\ \delta(k-1) + 0.2\ \delta(k-2) + 0.1\ \delta(k-3)$$

has a z-transform (also known as the moment generating function) of

$$\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$$

The assumption behind the z-transform is that all functions are in the form of

$$y(k) = z^k$$

When you move forward in time by one sample, you multiply by 'z'.

$$y(k+1) = z^{k+1} = z \cdot z^k = z \cdot y(k)$$

'zy' is read as 'the next value of y'. Likewise, the z-transform converts difference equations into algebraic equations in 'z'.

z-Transform Properties: (www.wikipedia.com)

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Properties:

Linearity:

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Proof: The z-transform is

$$Z(ax_n + by_n) = \sum_{n=-\infty}^{\infty} (ax_n + by_n) \cdot z^{-n}$$

$$Z(ax_n + by_n) = \left(a \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}\right) + \left(b \sum_{n=-\infty}^{\infty} y_n \cdot z^{-n}\right)$$

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Time Shifting:

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Proof:

$$Z(x_{n-k}) = \sum_{n=-\infty}^{\infty} x_{n-k} \cdot z^{-n}$$

Let m = n-k

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-(m+k)}$$

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \cdot z^{-k}$$

$$Z(x_{n-k}) = z^{-k} \cdot \left(\sum_{m=-\infty}^{\infty} x_m \cdot z^{-m}\right)$$

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Multiplying by 1/z means delay the signal by one.

Convolution:

$$Z(x_n ** y_n) = X(z) \cdot Y(z)$$

Proof:

$$Z\left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

Change the order of summation:

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

$$= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_{n-k}\right)\right) \cdot z^{-n}$$

Let m = n-k

$$= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_m\right)\right) \cdot z^{-(m+k)}$$

$$= \left(\left(\sum_{k=-\infty}^{\infty} x_k \cdot z^{-k}\right)\left(\sum_{n=-\infty}^{\infty} y_m \cdot z^{-m}\right)\right)$$

$$= X(z) \cdot Y(z)$$

Convolution is the biggie - it turns convolution into multiplication.

Table of z-Transforms:

function	y(k) (k > 0)	Y(z)
delta	$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$	1
unit step	$u(k) = 1$	$\left(\frac{z}{z-1}\right)$
ramp	k	$\left(\frac{z}{(z-1)^2}\right)$
parabola	k^2	$\left(\frac{z(z+1)}{(z-1)^3}\right)$
cubic	k^3	$\left(\frac{z(z^2+4z+1)}{(z-1)^4}\right)$
decaying exponential	a^k	$\left(\frac{z}{z-a}\right)$
	$\left(\frac{k}{1!}\right)a^{k-1}$	$\left(\frac{z}{(z-a)^2}\right)$
	$\left(\frac{k(k-1)}{2!}\right)a^{k-2}$	$\left(\frac{z}{(z-a)^3}\right)$
	$\left(\frac{k(k-1)(k-2)}{3!}\right)a^{k-3}$	$\left(\frac{z}{(z-a)^4}\right)$
damped sinewave	$2b \cdot a^k \cdot \cos(k\theta + \phi) \cdot u(k)$	$\left(\frac{(b\angle\phi)z}{z-(a\angle\theta)}\right) + \left(\frac{(b\angle-\phi)z}{z-(a\angle-\theta)}\right)$

Proof: Delta Function. This is sort-of the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \ldots + x_0 \cdot z^0 + x_1 \cdot z^1 + x_2 \cdot z^2 + \ldots$$

If x(k) is a delta function:

$$X(z) = 1$$

Proof: Unit Step. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	1	1	1	1
z^{-1} X(z)	0	0	0	1	1	1	1
subtract							
$\left(1 - \frac{1}{z}\right)X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \frac{1}{\left(1-\frac{1}{z}\right)} = \left(\frac{z}{z-1}\right)$$

Proof: Decaying Exponential. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
X(z)	0	0	1	a	a^2	a^3	a^4
a * z^{-1} X(z)	0	0	0	a	a^2	a^3	a^4
subtract							
$(1-\frac{a}{z})X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \left(\frac{1}{1-\frac{a}{z}}\right) = \left(\frac{z}{z-a}\right)$$

Solving Functions in the z-Domain

Problem 1: Find the step response of

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)X$$

i) Replace X(z) with the z-transform of a step

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

ii) Use partial fractions to expand this. Note, however, that the table entries have a 'z' in the numerator. So, factor this out first then take the partial fraction expansion

$$Y = \left(\frac{0.2z}{(z-1)(z-0.9)(z-0.5)}\right)z$$

$$Y = \left(\left(\frac{4}{z-1}\right) + \left(\frac{-4.5}{z-0.9}\right) + \left(\frac{0.5}{z-0.5}\right)\right)z$$

Multiply through by z

$$Y = \left(\left(\frac{4z}{z-1}\right) + \left(\frac{-4.5z}{z-0.9}\right) + \left(\frac{0.5z}{z-0.5}\right)\right)$$

iii) Now apply the table entries

$$y(k) = 4 - 4.5\cdot(0.9)^k + 0.5\cdot(0.5)^k \qquad k >= 0$$

Problem 2: Find the step response of a system with complex poles:

$$Y = \left(\frac{0.2z}{\left(z-0.9\angle 10^0\right)\left(z-0.9\angle -10^0\right)} \right) X$$

i) Replace X with its z-transforrm (a unit step)

$$Y = \left(\frac{0.2z}{\left(z-0.9\angle 10^0\right)\left(z-0.9\angle -10^0\right)} \right) \left(\frac{z}{z-1} \right)$$

ii) Factor our a z and use partial fractions

$$Y = \left(\left(\frac{5.355}{z-1} \right) + \left(\frac{2.98\angle 153.97^0}{z-0.9\angle 10^0} \right) + \left(\frac{2.98\angle -153.97^0}{z-0.9\angle -10^0} \right) \right) z$$

iii) Convert back to time using the table of z-transforms

$$(k) = 5.355 + 4.859 \cdot (0.9)^k \cdot \cos\left(10^0 \cdot k - 153.97^0\right) \qquad k >= 0$$

Note that in the s-domain, rectangular coordinates are more convenient

- The real part of s tells you the rate at which the exponential decays
- The complex part of s tells you the frequency of oscillations.

In the z-domain, polar coordinates are more convenient

- The amplitude of z tells you the rate at which the signal decays
- The angle of z tells you the frequency of oscillation.

In this case,

- The signal decays by 10% each sample (0.9)k
- The phase changes by 10 degrees each sample, or requires 36 samples per cycle

Time Value of Money

As a sidelight, you can also solve time-value of money problems using z-transforms.

Assume you borrow $100,000 for a house. How much do you have to pay each month to pay off the loan in 10 years? Assume 6% interest per year (0.5% per month).

Solution: Let x(k) be how much money you owe today. The amount you owe next month, x(k+1), is

$$x(k+1) = 1.005x(k) - p + X(0) \cdot \delta(k)$$

where 'p' is your monthly payment. Converting to the z-domain

$$zX - X(0) = 1.005X - p\left(\frac{z}{z-1}\right)$$

$$(z-1.005)X = X(0) - p\left(\frac{z}{z-1}\right)$$

$$X = \left(\frac{X(0)}{z-1.005}\right) - p\left(\frac{z}{(z-1)(z-1.005)}\right)$$

Using partial fractions

$$X = \left(\frac{X(0)}{z-1.005}\right) + pz\left(\left(\frac{200}{z-1}\right) - \left(\frac{200}{z-1.005}\right)\right)$$

Converting back to the time domain

$$x(k) = 1.005^k X(0) - 200p(1.005^k - 1)u(k)$$

After 10 years (k=121 payments), x(k) should be zero (you make 120 payments, meaning at k=121, your balance is zero, meaning you owe no more money)

$$x(121) = 0 = \$182,849 - 200p(0.8285)$$

$$p = \$1103.50$$

Your monthly payments are $1,103.50.

If you stretch this out to 30 years (k = 360 payments), the monthly payment becomes

$$x(361) = 0 = \$605,268 - 200p(5.0527)$$

$$p = \$598.57$$

Paying off the loan over a time span 3 times longer doesn't reduce the payments by 3 times. It's actually only 46% less. The total amount you'll pay on the loan, however, increases from $133,224 to $215,838.

note: That's pretty much all a business calculator is: a calculator which does z-transforms where the keys are renamed "interest rate", "initial loan value" and "number of payments."