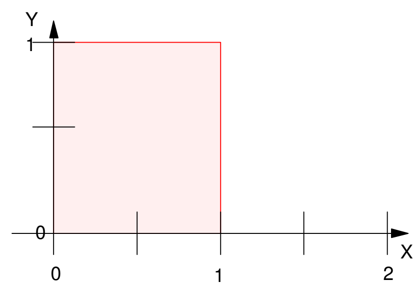
Multiple Continuous Random Variables

Problem: Define the pdf and CDF for a funciton of 2 or more random variables.

Example: Let Z(X,Y) be the point on the X-Y plane where X and Y are independent uniformly distributed random variables on the interval (0,1).

The range of Z is the unit box:



Joint CDF:

$$F_{X,Y}(x,y) = P[X < x, Y < y]$$

From this definition, we get

$$F_X(x) = P[X < x]$$

(y doesn't matter), so

$$F_X(x) = P[X < x, Y < \infty]$$

$$F_X(x) = F_{X,Y}(x, \infty)$$

Properties of joint CDF:

- $0 \le F_{X,Y}(x,y) \le 1$
- $F_X(x) = F_{X,Y}(x, \infty)$
- $F_Y(y) = F_{X,Y}(\infty, y)$
- $F_{X,Y}(x, -\infty) = F_{X,Y}(-\infty, y) = 0$
- $F_{X,Y}(\infty, \infty) = 1$

Joint PDF:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) \cdot du \cdot dv$$

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(F_{X,Y}(x,y))\right)$$

note about partial derivatives:

$\frac{\partial}{\partial x}(F)$ is the partial derivarive of F with respect to x

$\frac{d}{dx}(F)$ is the full derivative of F with respect to x.

The two are different. F

- When taking the partial derivative with respect to x, treat everything except 'x' as a constant.
- When taking the full derivative with respect to x, everything can vary.

For example, consider

$$f(t,x,y) = 3t^2 + x^2y^3$$

$$x = 4t^2 \qquad \left(\frac{dx}{dt} = 8t\right)$$

$$y = 5t^3 \qquad \left(\frac{dy}{dt} = 15t^2\right)$$

Find the partial derivative of f() with respect to t, x, and y:

$$\frac{\partial}{\partial t}(3t^2 + x^2y^3) = 6t \qquad \text{(x and y are treated as constants)}$$

$$\frac{\partial}{\partial x}(3t^2 + x^2y^3) = 2xy^3 \qquad \text{(t and y are treated as constants)}$$

$$\frac{\partial}{\partial y}(3t^2 + x^2y^3) = 3x^2y^2 \qquad \text{(t and x are teated as constants)}$$

Find the full derivative of f() with respect to t. There are three terms

- The change in f() with respect to t, plus
- The change in f() with respect to x (the slope in the x direction) times how fast x is changing, plus
- The change in f() with respect to y (the slope in the y direction) times how fast y is changing:

$$\frac{d}{dt}(f()) = \left(\frac{\partial f()}{\partial t}\right) + \left(\frac{\partial f()}{\partial x}\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f()}{\partial y}\frac{\partial y}{\partial t}\right)$$

Note that partial derivatives are used to compute a full derivative (i.e. they are different.) The reason we treat everything except x as constant when taking the partial with respec to x is we don't want to double count the change in y or t.

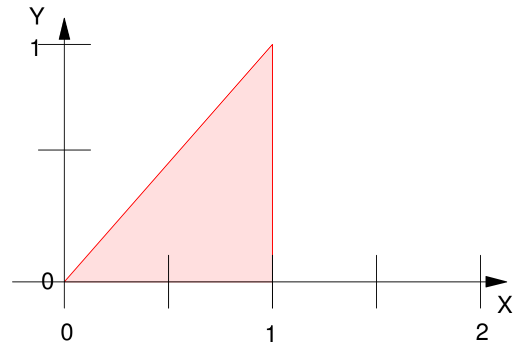
Continuing on...

$$\frac{d}{dt}(f()) = (6t) + \left(2xy^3 \cdot \frac{dx}{dt}\right) + \left(3x^2y^2\frac{dy}{dt}\right)$$

You could substitute for x, y, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ if you like at this point...

Anyway, back to joint PDF's.

Suppose all points in the following shaded region are equally likely:



Find the joint CDF and PDF.

First, the volume must be one (think of p(x,y) being the height of this 3-D figure.) The volume is base * height. The base has an area of 1/2, so the height must be 2. You can generate this shape noting

- $0 < y < x$
- $y < x < 1$

or

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & otherwise \end{cases}$$

To find the joint CDF, integrate:

$$F_{X,Y}(x,y) = \int_0^y \left(\int_v^x 2 \cdot du \right) dv$$

$$= \int_0^y \left((2u)_{u=v}^{u=x} \right) dv$$

$$= \int_0^y (2x - 2v) dv$$

$$= (2xv - v^2)_{v=0}^{v=y}$$

$$= (2xy - y^2) - 0$$

$$F_{X,Y}(x,y) = (2xy - y^2) \qquad 0 < y < x < 1$$

Note that the range and uniform probability over the area is easy to see in the PDF. The CDF hides this information.

As a check....

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(2xy - y^2)\right) \qquad 0 < y < x < 1$$

$$= \frac{\partial}{\partial x}(2x - 2y)$$

$$= 2 \qquad 0 < y < x < 1$$

Marginal PDF

Given a joint PDF for x and y, you can generate the PDF for x alone as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \cdot dy$$

Example: Given the previous PDF for x and y, determine the PDF for x and the PDF for y:

$$f_{X,Y}(x,y) = 2 \qquad 0 < y < x < 1$$

$$f_X(x) = \int_0^x 2 \cdot dy$$

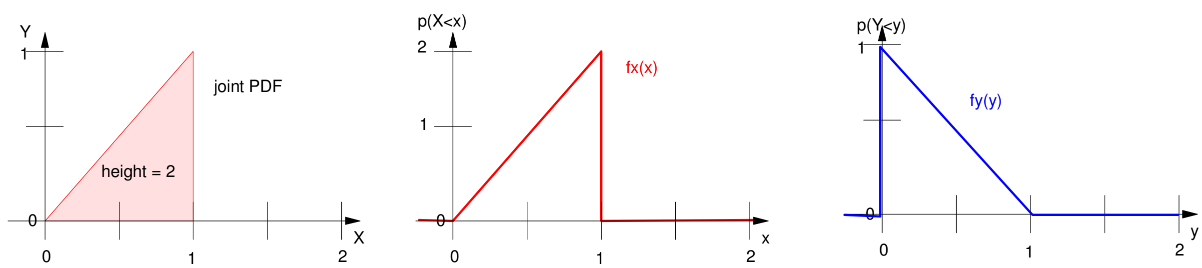
$$= (2y)_{y=0}^{y=x}$$

$$f_X(x) = 2x \qquad 0 < x < 1$$

$$f_Y(y) = \int_y^1 2 \cdot dx$$

$$= (2x)_{x=y}^{x=1}$$

$$f_Y(y) = 2(1 - y) \qquad 0 < y < 1$$



Non-Uniform Joint PDF

Just to make life interesting (and to show off some functions in MATLAB), consider a PDF with a variable probability.

First, let's plot the PDF for the previous case. A function in MATLAB which does this is mesh().

```
mesh(z)
mesh(x,y,z)
```

mesh() draws a 3D plot where z is an nxm array whose values are the height of the function. For the previous case:

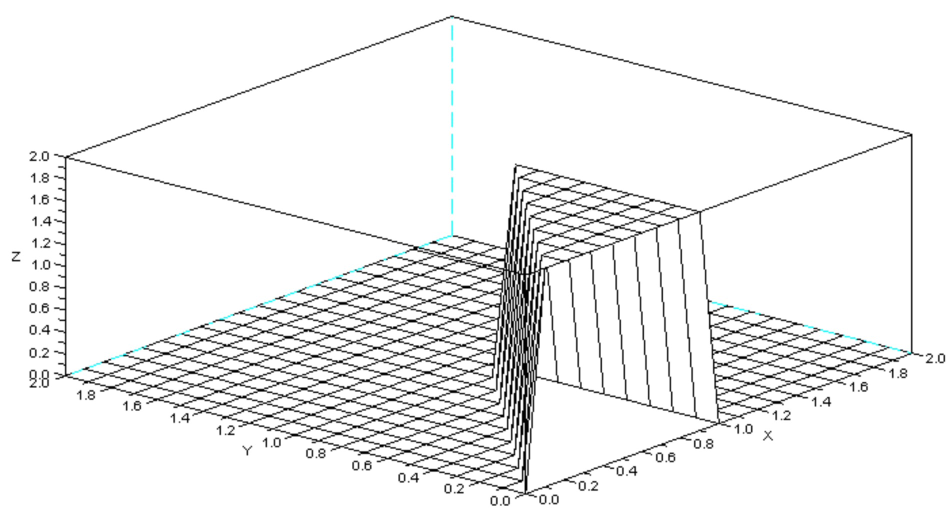
```
x = [0:0.1:2]';
y = [0:0.1:2]';

N = length(x);

z = zeros(N,N);

for i=1:21
   for j=1:21
      if (x(i) <= 1)
         if (y(j) <= x(i))
            z(j,i) = 2;
            end
         end
      end
   end

mesh(x,y,z)
```



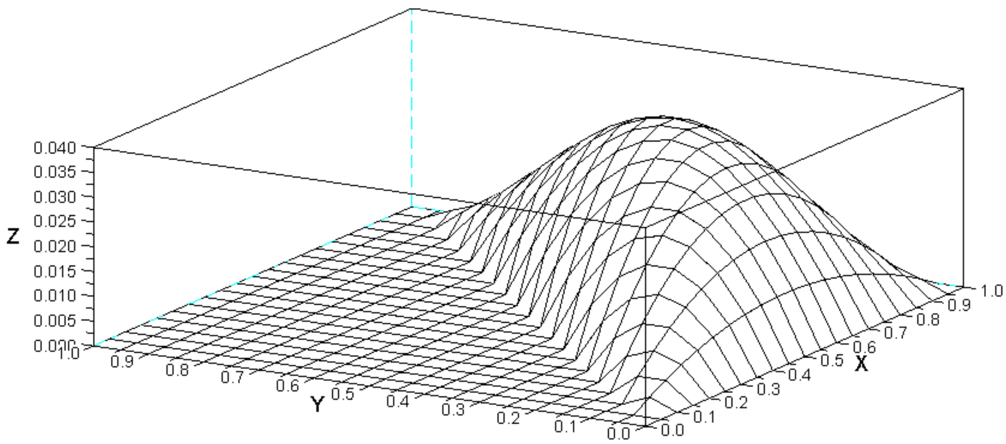
Checking, the volume should be one:

```
sum(z) * 0.1 * 0.1 = 1.32
```

It's a little off due to the course grid and using <= for x and y.

Let's try this with a non-flat surface:

$$f_{X,Y}(x,y) = \alpha \cdot (1-x) \cdot y \cdot (x-y) \qquad\qquad 0 < y < x < 1$$



α is a constant to make the volume equal to one.

Find the CDF:

$$F_{X,Y}(x,y) = \int_0^y \left(\int_v^x \alpha \cdot (1-u) \cdot v \cdot (u-v) \cdot du \right) dv$$

$$F_{X,Y}(x,y) = \int_0^y \left(\int_v^x \alpha \cdot (uv - u^2v - v^2 + uv^2) \cdot du \right) dv$$

$$= \alpha \int_0^y \left((\tfrac{1}{2}u^2v - \tfrac{1}{3}u^3v - uv^2 + \tfrac{1}{2}u^2v^2)_{u=v}^{u=x} \right) dv$$

$$= \alpha \int_0^y \left((\tfrac{1}{2}x^2v - \tfrac{1}{3}x^3v - xv^2 + \tfrac{1}{2}x^2v^2) - \left(\tfrac{1}{2}v^3 - \tfrac{1}{3}v^4 - v^3 + \tfrac{1}{2}v^4 \right) \right) dv$$

$$= \alpha \int_0^y \left(\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right) v + \left(\tfrac{1}{2}x^2 - x \right) v^2 + \tfrac{1}{2}v^3 - \tfrac{1}{6}v^4 \right) dv$$

$$= \alpha \left(\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right) \tfrac{1}{2}v^2 + \left(\tfrac{1}{2}x^2 - x \right) \tfrac{1}{3}v^3 + \tfrac{1}{8}v^4 - \tfrac{1}{30}v^5 \right)_{v=0}^{v=y}$$

$$F_{X,Y}(x,y) = \alpha \left(\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right) \tfrac{1}{2}y^2 + \left(\tfrac{1}{2}x^2 - x \right) \tfrac{1}{3}y^3 + \tfrac{1}{8}y^4 - \tfrac{1}{30}y^5 \right)$$

Now, let's find the marginal PDF with respect to x:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \cdot dy$$

$$f_X(x) = \alpha \int_0^x (1-x) \cdot y \cdot (x-y) \cdot dy$$

$$= \alpha(1-x)\int_0^x (xy - y^2) \cdot dy$$

$$= \alpha(1-x)\left(\tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3\right)_{y=0}^{y=x}$$

$$= \alpha(1-x)\left(\tfrac{1}{2}x^3 - \tfrac{1}{3}x^3\right)$$

$$f_X(x) = \tfrac{\alpha}{6}(1-x)x^3 \qquad 0 < x < 1$$

α is a constant to make the area one (i.e. make this a valid PDF)

Plotting f(x) in MATLAB:

```
-->x = [0:0.001:1]';

-->f = (1-x) .* (x .^ 3);
```

Scale so that the area is one (i.e. so that this is a valie PDF)

```
-->sum(f) * 0.001
    0.0499999

-->1/ans
    20.000033
```

To be a valid PDF,

$$f_X(x) = 20(1-x)x^3 \qquad 0 < x < 1$$

```
-->f = 20 * (1-x) .* (x .^ 3);
-->plot(x,f)
-->xlabel('x');
-->ylabel('f(x)');
```