LaPlace Transforms

ECE 341: Random Processes

Lecture #10

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & otherwise \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer functions and back.

Example 1: Find the transfer function from X to Y

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

The transfer function from X to Y is

$$\boxed{G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)}$$

- Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Find the differential equation relating X and Y

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

Cross multiply:

$$(s^3+6s^2+11s+6)Y = (8s+10)X$$

Note that 'sY' means 'the derivative of Y'

$$\boxed{\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x}$$

Sidelight: Fractional powers are not allowed in transfer functions.

- $s^2 Y$ means 'the second derivative of Y'.
- $s^{2.3} Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find y(t) given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

and

$$x(t) = 3\cos(4t)$$

Solution: From Euler's identity:

$$e^{j4t} = \cos(4t) + j\sin(4t)$$

$$(a+jb)\,e^{j4t} = (a\cos(4t) - b\sin(4t)) + j(\cdot)$$

$$a+jb \leftrightarrow a\cos(4t) - b\sin(4t)$$

- real = cosine
- -imag = sine

Take the real part and you the sine wave.

$$s = j4$$

$$X = 3 + j0$$

Convert to phasors

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right) X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j4} (3 + j0)$$

$$Y = (-0.191 - j0.315)(3 + j0)$$

$$Y = -0.544 - j0.946$$

meaning

$$y(t) = -0.544\cos(4t) + 0.946\sin(4t)$$

Example 4: Find y(t) if

$$x(t) = 5\cos(20t) + 6\sin(20t)$$

Solution:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j20}(5 - j6)$$

$$Y = (-0.019 - j0.005)(5 - j6)$$

$$Y = -0.123 + j0.092$$

meaning

$$y(t) = -0.123\cos(20t) - 0.092\sin(20t)$$

Note that the gain varies with frequency (i.e. this is a filter).

Example 5: Find y(t) if

$$x(t) = 3\cos(4t) + 5\cos(20t) + 6\sin(20t)$$

Solution: Treat this as two separate problems

- $x(t) = 3\cos(4t)$
- $x(t) = 5\cos(20t) + 6\sin(20t)$

The total input is the sum of the two x(t)'s.

The total output is the sum of the two y(t)'s

$$y(t) = -0.544\cos(4t) + 0.946\sin(4t)$$
$$-0.123\cos(20t) - 0.092\sin(20t)$$

Solving Transfer Functions with Step Inputs

Common LaPlace Transforms		
Name	Time: y(t) (t >0)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	1	$\left(\frac{1}{s}\right)$
unit ramp	t	$\left(\frac{1}{s^2}\right)$
unit parabola	t^2	$\left(\frac{2}{s^3}\right)$
decaying exponential	e^{-bt}	$\left(\frac{1}{s+b}\right)$
	$t\ e^{-bt}$	$\left(\frac{1}{(s+b)^2}\right)$
	$t^2\ e^{-bt}$	$\left(\frac{2}{(s+b)^3}\right)$
	$2a \cdot e^{-bt}\cos(ct - \theta)u($	$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right) X$$

$$Y = \left(\frac{5}{s+3}\right)(1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$

Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Do a partial fraction expansion

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

Using the above table for each term:

$$\boxed{y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)}$$

Repeated Roots

From the table of LaPlace Transforms

$$\left(\frac{(n-1)!}{(s+a)^n}\right) \leftrightarrow t^n e^{-at} u(t)$$

The highest power can be found using partial fractions and the cover-up method

The other terms can be found by using a common denominator along with algebra

Example 1: Find the inverse-LaPlace transform of

$$Y = \left(\frac{4}{s(s+2)^2}\right)$$

You know the answer is in the form of

$$\left(\frac{4}{s(s+2)^2}\right) = \left(\frac{a}{s}\right) + \left(\frac{b}{(s+2)^2}\right) + \left(\frac{c}{s+2}\right)$$

Find a and b using the cover-up method

$$a = \left(\frac{4}{(s+2)^2}\right)_{s=0} = 1$$

$$b = \left(\frac{4}{s}\right)_{s=-2} = -2$$

Find c by using a common denominator and matching terms

$$\left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{c}{s+2}\right)$$

$$\left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right)\left(\frac{(s+2)^2}{(s+2)^2}\right) + \left(\frac{-2}{(s+2)^2}\right)\left(\frac{s}{s}\right) + \left(\frac{c}{s+2}\right)\left(\frac{s(s+2)}{s(s+2)}\right)$$

$$4 = (s+2)^2 - 2s + cs(s+2)$$

Match the s^2 term

$$0s^2 = s^2 + cs^2 \qquad \Rightarrow \qquad c = -1$$

So

$$Y = \left(\frac{4}{s(s+2)^2}\right) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{-1}{s+2}\right)$$

$$y(t) = (1 - 2te^{-2t} - e^{-t})u(t)$$

Example 2: Find y(t)

$$Y = \left(\frac{8}{s(s+2)^3}\right)$$

Use partial fractions

$$Y = \left(\frac{a}{s}\right) + \left(\frac{b}{(s+2)^3}\right) + \left(\frac{c}{(s+2)^2}\right) + \left(\frac{d}{s+2}\right)$$

a and b can be found using the cover-up method

$$a = \left(\frac{8}{(s+2)^3}\right)_{s=0} = 1$$

$$b = \left(\frac{8}{s}\right)_{s=-2} = -4$$

To find c and d, place over a common denominator and match terms

$$\left(\frac{8}{s(s+2)^3}\right) = (\tfrac{a}{s}) + \left(\frac{b}{(s+2)^3}\right) + \left(\frac{c}{(s+2)^2}\right) + \left(\frac{d}{s+2}\right)$$

$$\left(\frac{8}{s(s+2)^3}\right) = (\tfrac{a}{s})\left(\frac{(s+2)^3}{(s+2)^3}\right) + \left(\frac{b}{(s+2)^3}\right)(\tfrac{s}{s}) + \left(\frac{c}{(s+2)^2}\right)\left(\frac{s(s+2)}{s(s+2)}\right) + \left(\frac{d}{s+2}\right)\left(\frac{s(s+2)^2}{s(s+2)^2}\right)$$

Match the numerator

$$8 = a(s+2)^3 + bs + cs(s+2) + ds(s+2)^2$$

s^3 term:

$$0s^3 = as^3 + ds^3 \qquad\qquad d = -a = -1$$

s^2 term:

$$0s^2 = 6as^2 + cs^2 + 4ds^2$$

$$0 = 6a + c + 4d \qquad\qquad c = -2$$

Result:

$$Y = \left(\frac{8}{s(s+2)^3}\right) = \left(\frac{1}{s}\right) + \left(\frac{-4}{(s+2)^3}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{-1}{s+2}\right)$$

Take the inverse LaPlace transform

$$y(t) = (1 - 2t^2 e^{-2t} - 2te^{-2t} - e^{-2t})u(t)$$

Alternate Method: Change the problem

$$Y = \left(\frac{8}{s(s+2)^3}\right) \approx \left(\frac{8}{s(s+1.99)(s+2.00)(s+2.01)}\right)$$

Now do partial fractions

$$Y = \left(\frac{a}{s}\right) + \left(\frac{b}{s+1.99}\right) + \left(\frac{c}{s+2.00}\right) + \left(\frac{d}{s+2.01}\right)$$

$$Y = \left(\frac{1}{s}\right) + \left(\frac{-20{,}100.5025}{s+1.99}\right) + \left(\frac{40{,}000}{s+2}\right) + \left(\frac{-19{,}900.4975}{s+2.01}\right)$$

Take the inverse LaPlace transform

$$y(t) = (1 - 20,100.5025e^{-1.99t} + 40,000e^{-2t} - 19,900.4975e^{-2.01t})u(t)$$

Comparing Results

- Change the problem gives answers that are good to four decimal places

t	y(t) (exact)	y(t) (approximate)
0	0	0
2.50	0.875347980516919	0.875337136234890
5.00	0.997230604284489	0.997229874022671
7.50	0.999960691551816	0.999960670138687
10.00	0.999999544485050	0.999999544061323

Solving with Complex Roots:

$$\left(\frac{a\angle\theta}{s+b+jc}\right)+\left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a\cdot e^{-bt}\cos(ct-\theta)u(t)$$

Example: Find the y(t) given that

$$Y(s) = G\cdot U = \left(\frac{15}{s^2+2s+10}\right)\cdot\left(\frac{1}{s}\right)$$

Solution: Factoring Y(s)

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right)+\left(\frac{0.7906\angle-161.56^0}{s+1+j3}\right)+\left(\frac{0.7906\angle161.56^0}{s+1-j3}\right)$$

$y(t) = 1.5 + 1.5812\cdot e^{-t}\cdot\cos\left(3t+161.56^0\right)$ for t>0

Properties of LaPlace Transforms

Linearity: $af(t)+bg(t) \Leftrightarrow aF(s)+bG(s)$

Convolution: $f(t)**g(t) \Leftrightarrow F(s)\cdot G(s)$

Differentiation:

$$\frac{dy}{dt} \Leftrightarrow sY-y(0)$$

$$\frac{d^2y}{dt^2} \Leftrightarrow s^2Y-sy(0)-\frac{dy(0)}{dt}$$

Integration: $\int_0^t x(\tau)d\tau = \frac{1}{s}X(s)$

Delay $x(t-T) \Leftrightarrow e^{-sT}X(s)$

Proofs

Linearity:

$$L(af(t)+bg(t)) = \int_{-\infty}^{\infty} (af(t)+bg(t)) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt$$

$$= a\int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt$$

$$= aF(s)+bG(s)$$

Convolution:

$$f(t) * *g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau$$

$$L(f(t) * *g(t)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau$$

$$= \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot e^{-st} \cdot dt \right) \cdot \left(\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

Differentiation:

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for t<0)

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$
\begin{aligned}
L\left(\frac{dx}{dt}\right) &= \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt \\
&= \left(x \cdot e^{-st}\right)_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt \\
&= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\
&= sX - x(0)
\end{aligned}
$$

Integration:

$$L\left(\int_0^t x(\tau)\cdot d\tau\right)=\int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

Integrate by parts.

$$\int a\cdot b'\cdot dt=ab-\int a'\cdot b\cdot dt$$

Let

$$a=\int_0^t x(\tau)\cdot d\tau$$

$$b'=e^{-st}$$

then

$$a'=x$$

$$b=\tfrac{-1}{s}e^{-st}$$

$$L\left(\int_0^t x(\tau)\cdot d\tau\right)=\int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

$$= \left(\int_0^t x(\tau) \cdot d\tau \cdot \frac{-1}{s} e^{-st} \right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \cdot \frac{-1}{s} e^{-st} \cdot dt$$

Assuming the function vanishes at infinity

$$= \frac{1}{s} \int_{-\infty}^{\infty} x \cdot dt$$

$$= \left(\frac{1}{s} \right) X(s)$$

Time Delay

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(t-T) \cdot e^{-st} \cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s(\tau+T)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot e^{-sT} \cdot d\tau$$

$$= e^{-sT} \cdot \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot d\tau$$

$$= e^{-sT} \cdot X(s)$$

Summary

LaPlace transforms

- Turn differential equations into algebraic equations
- Turn convolution in to multiplication

They are very useful in solving differential equations

- i.e. Analog Signals
- Multiplication is easier than convolution

They will also be useful in Random Processes

- If we ever need to convolve continuous pdf's, LaPlace transforms will turn this into multiplication
- In statistics, LaPlace transforms are termed *moment generating functions*