Exponential Distribution

ECE 341: Random Processes

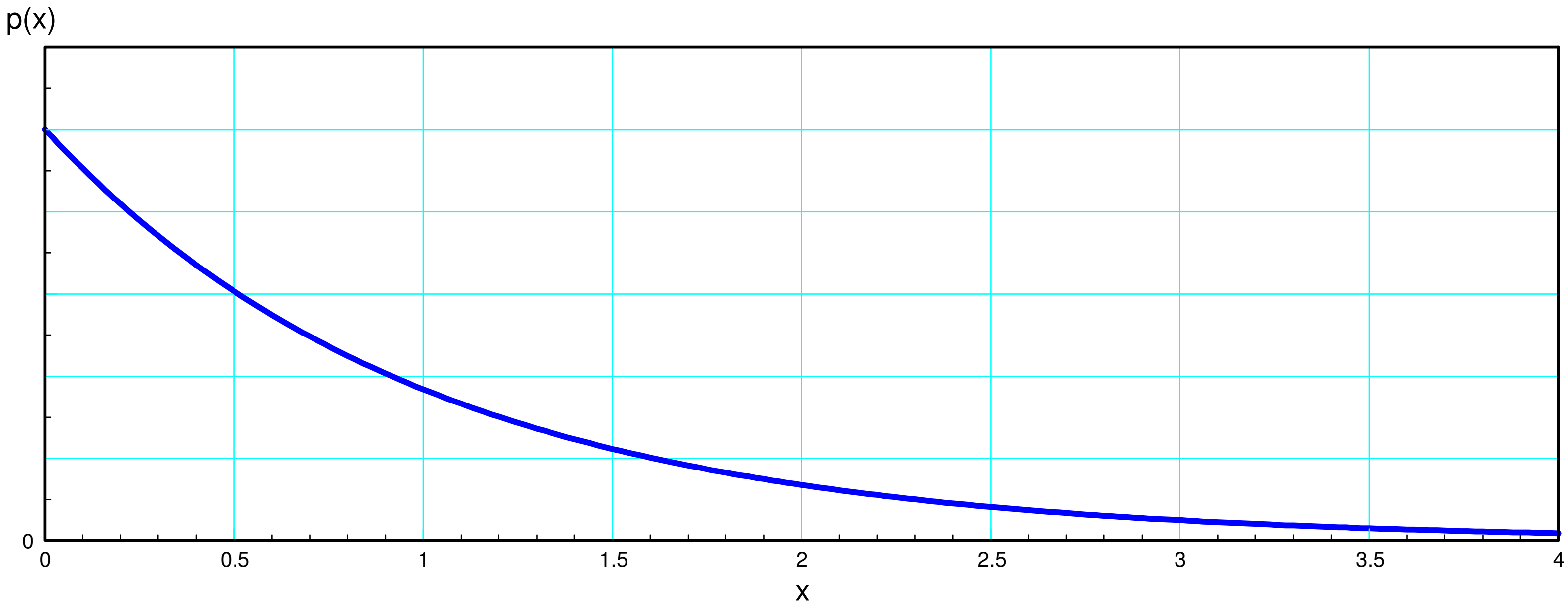
Lecture #13

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Exponential Distribution

Another common continuous probabilty distribution is the exponential distribution. With this distribution, the pdf is exponential:

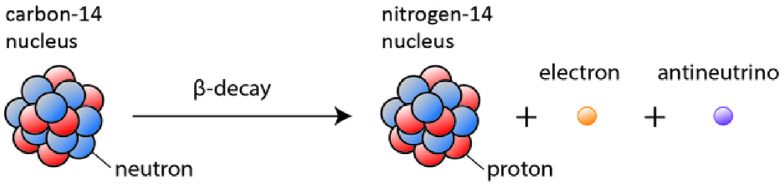
$$f_X(x) = \begin{cases} a\, e^{-ax} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$



pdf for an exponential distribution

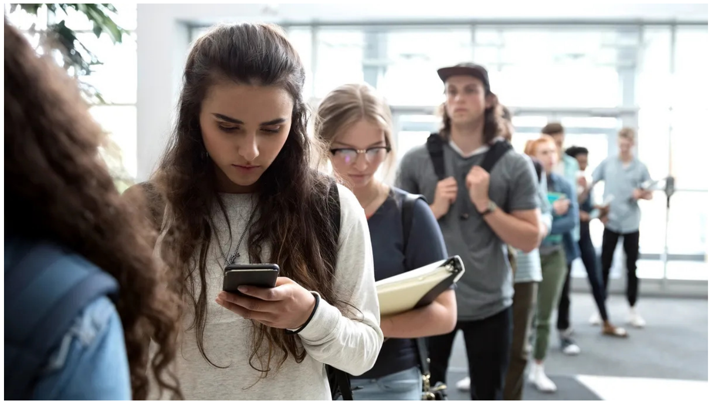
Examples of where this type of distibution is encountered is:

- Probability that the length of a telephone call is less than x minutes
- Probability that the next atom will decay within x seconds
- Time it takes for the next customer to arrive at a store
- Time it takes to serve the next customer



Exponential distributions also lead into queueing theory:

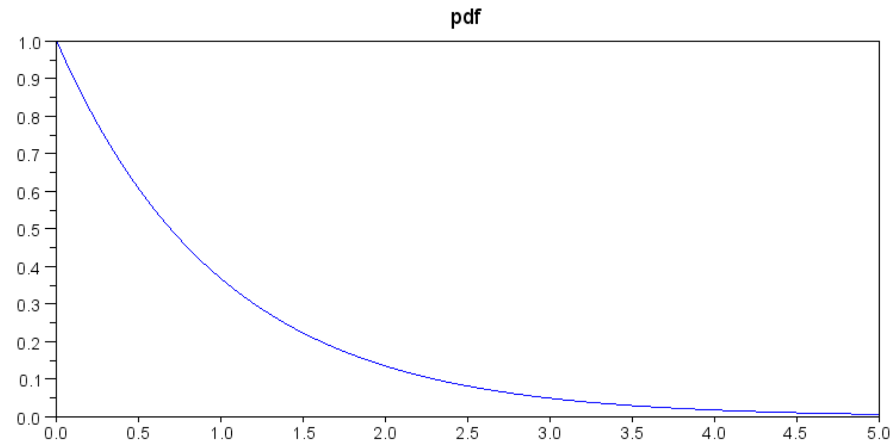
- How long a customer will have to wait in line to be served.



Parameters of Exponential Distributions

pdf: As stated before

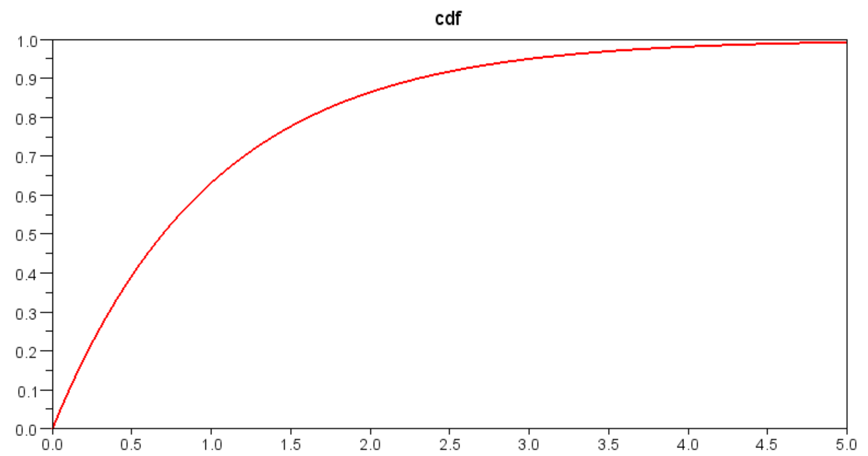
$$pdf(x) = a\, e^{-ax}$$



cdf: The cdf is the integral of the pdf

$$cdf(x) = \int_0^x (a\, e^{-at})\, dt$$

$$cdf(x) = 1 - e^{-ax} \qquad x > 0$$



Moment Generating Function:

$$\psi(s) = \left(\frac{a}{s+a}\right)$$

To be a valid probability distribution:

$$m_0 = \psi(s = 0) = 1 = (\tfrac{a}{a})$$

Mean: The mean of an exponential distribution is

$$\mu = \int_0^\infty p(x)\, x\, dx = \int_0^\infty (ae^{-ax})\, x\, dx = \left(\frac{1}{a}e^{-ax}(-ax - 1)\right)\Bigg|_0^\infty = \left(\frac{1}{a}\right)$$

$$\mu = m_1 = -\psi'(0) = \left(\frac{a}{(s+a)^2}\right)\Bigg|_{s=0} = \frac{1}{a}$$

Variance: Use the moment generating function - it's a lot easier

$$m_2 = -\psi''(0) = -\frac{d}{ds}\left(\frac{a}{(s+a)^2}\right)$$

$$m_2 = -\frac{d}{ds}\left(\frac{a}{(s+a)^2}\right)_{s=0} = \left(\frac{2a}{(s+a)^3}\right)_{s=0} = \left(\frac{2}{a^2}\right)$$

so

$$\sigma^2 = m_2 - m_1^2$$

$$\sigma^2 = \left(\frac{2}{a^2}\right) - \left(\frac{1}{a}\right)^2 = \left(\frac{1}{a^2}\right)$$

Matlab Example:

Use the cdf:

$$cdf(x) = \int_0^x pdf(t)\, dt = (1 - e^{-ax})u(x)$$

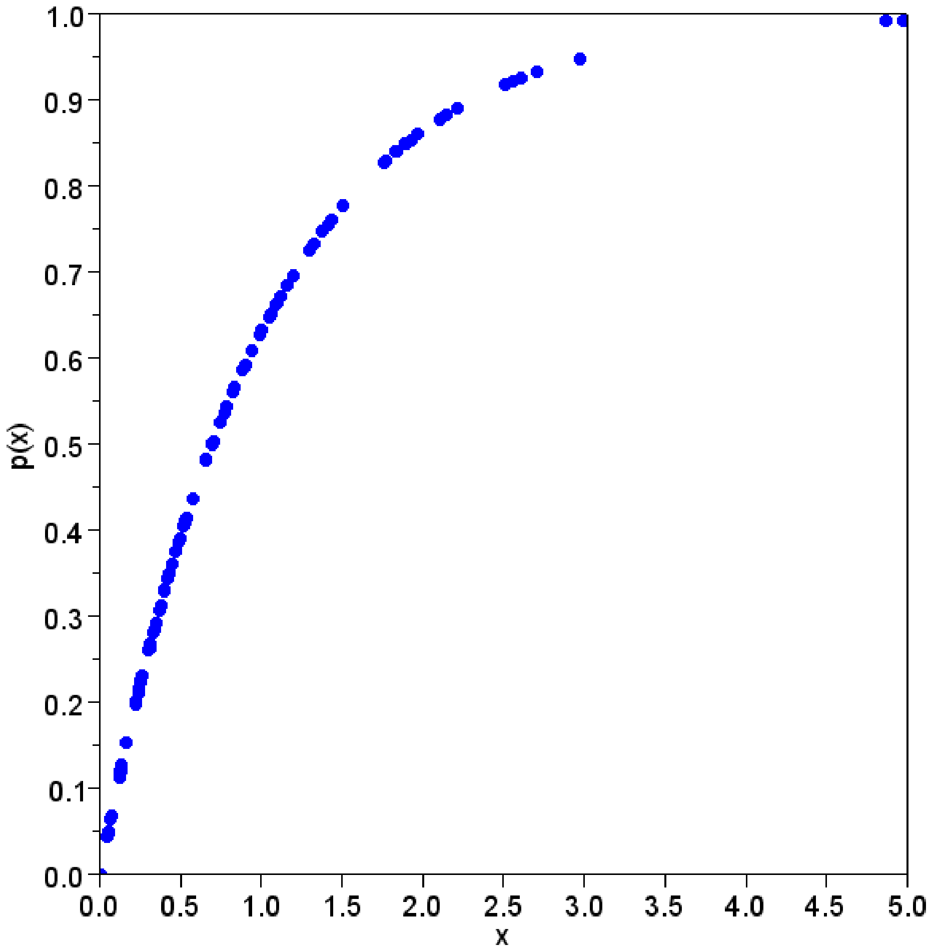
$$x = -\left(\frac{1}{a}\right) \ln(1 - p)$$

In matlab

```
p = rand(10,1);

a = 1;
x = -(1/a) * log(1-p)
```

p	x
0.6862	1.1590
0.9593	3.2004
0.4850	0.6635
0.7880	1.5511
0.7550	1.4066
0.7228	1.2831
0.2646	0.3074
0.6884	1.1660
0.9323	2.6930



Example: A block of radioactive material is sitting next to a Geiger counter. In 30 minutes, the Geiger counter detects 100 atoms decaying. Determine the pdf and CDF for the time until the next atom decays (in minutes).

Solution: The average number of atoms decaying per minute is

$$\bar{x} = \frac{100 \text{ atoms}}{30 \text{ minutes}} = 3.33 \frac{\text{atoms}}{\text{min}} = \frac{1}{a}$$

$$a = 0.3$$

pdf:

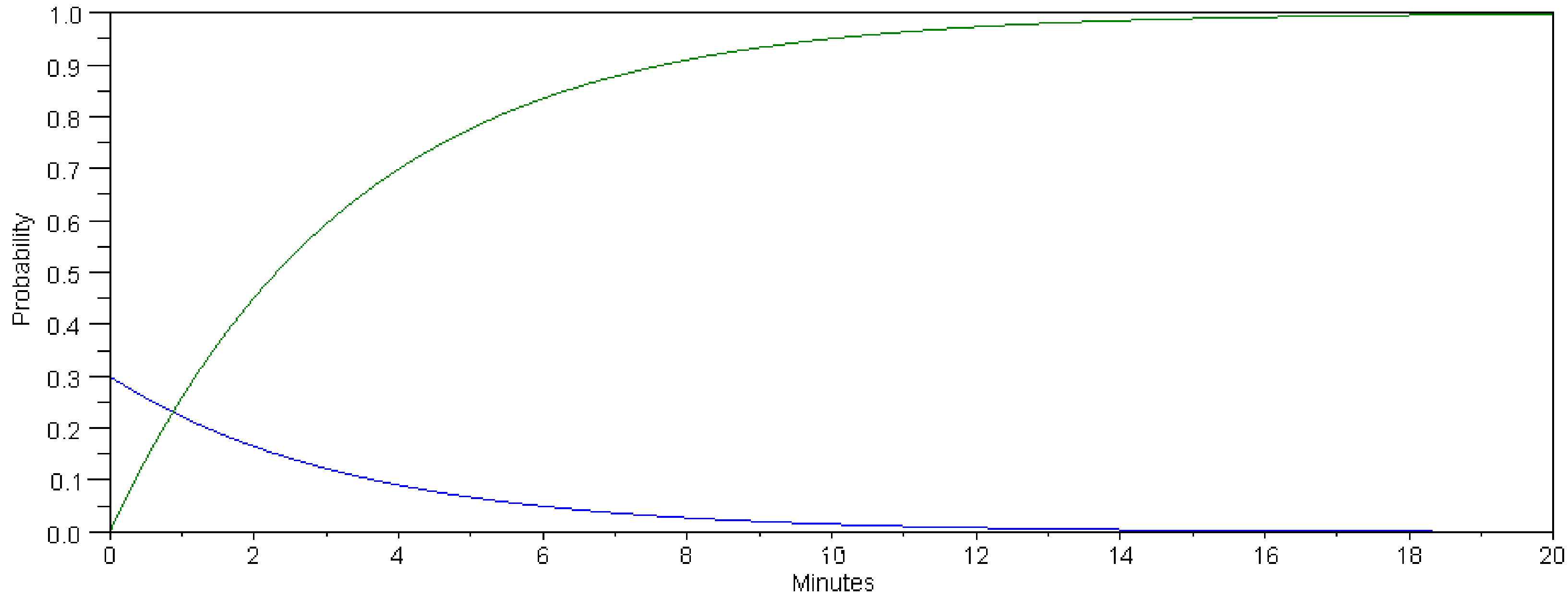
$$f_X(x) = \begin{cases} 0.3 \cdot e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

CDF:

$$F_X(x) = \begin{cases} 1 - e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Matlab

```
x = [0:0.01:20]';
p = 0.3 * exp(-0.3*x);
C = 1 - exp(-0.3*x);
plot(x,p,x,C)
xlabel('Minutes');
ylabel('Probability');
```



pdf (blue) and CDF (green) for time until you detect an atom decaying

Queueing Theory

One place where exponential distributions are often used are in determining how many servers are needed at a restaraunt.

For example, assume

- Customers arrive at a fast-food restaraunt with
  - An exponential distribution, with
  - An average of one customer every minute.
- It takes 30 seconds to serve each customer

How long will the longest wait be for a 1 hour shift?

Start with generating 1 hours worth of custumers:

```
a = 1/60;
s = 0;
x = [];
Tarr = []
n = 0;
t = 0;

while(t < 3600)
    n = n + 1;
    p = rand;
    x = -(1/a)*log(1-p);
    t = t + x;
    TIME(n) = t;
    disp([n, p, x, t])
end
```

58 customers arrived over this hour with arrival times.

Command Window

```
     []

    1.0000    0.1689   11.0989   11.0989

    2.0000    0.7452   82.0287   93.1276

    3.0000    0.4771   38.9058  132.0334

    4.0000    0.6534   63.5827  195.6162

    5.0000    0.9666  203.9087  399.5249

    6.0000    0.3130   22.5277  422.0526

    7.0000    0.0764    4.7711  426.8237

    8.0000    0.7914   94.0446  520.8683

    9.0000    0.3654   27.2841  548.1524

   10.0000    0.5851   52.7830  600.9354

   11.0000    0.1833   12.1517  613.0871

   12.0000    0.0769    4.8023  617.8894

   13.0000    0.1537   10.0102  627.8996

   14.0000    0.8269  105.2248  733.1244
```

As each customer arrives, their service time is the next available slot:

- 30 seconds after the last custimer is served, or
- Immediately if there is no wait.

The wait time is the difference in time from being served to time of arrival

The max queue size is the number of customers waiting to be served at the time you arrive (their finish time is more than your arrival time)

	a	b	c	d	e	f
1	**Customer**	**Arrival Time**	**Serve Time**	**Finish Time**	**Wait Time**	**Queue Size**
2	1	33	33	63	0	0
3	2	63	63	93	0	0
4	3	70	93	123	23	1
5	4	89	123	153	34	2
	=a4+1	paste from Matlab	=max(b5,d4)	=c5+30	=c5-b5	=1*(d4>b5) + 1*(d3>b5) + 1*(d2>b5)
6	5	118	153	183	35	2
7	6	421	421	451	0	0
8	7	454	454	484	0	0
9	8	473	484	514	11	1
10	9	476	514	544	38	2
11	10	476	544	574	68	3
12	11	477	574	604	97	4
13	12	484	604	634	120	4
14	13	544	634	664	90	3
15	14	547	664	694	117	4
16	15	692	694	724	2	1

What you want is

- To have the fewest servers needed, while
- Keeping the wait time less than some threshold (such as 2 minutes. If a customer has to wait more than X about of time, they'll leave)
- Keeping the queue size less than some threshold (if too many people are in line, customers will leave).

Customer	Arrival Time	Serve Time	Finish Time	Wait Time	Queue Size
1	33	33	63	0	0
2	63	63	93	0	0
3	70	93	123	23	1
4	89	123	153	34	2
5	118	153	183	35	2
6	421	421	451	0	0
7	454	454	484	0	0
8	473	484	514	11	1
9	476	514	544	38	2
10	476	544	574	68	3
11	477	574	604	97	4
12	484	604	634	120	4
13	544	634	664	90	3
14	547	664	694	117	4
15	692	694	724	2	1

Summary

Exponential distributions model many systems

- Time of a phone call
- Time until an atom decays
- Time until the next customer arrives

Queeing theory is based upon exponential distributions

- Used to determine expected wait times queue sizes in stores and restaraunts

If you want to know the time until 5 customers arrive, that's a different distribution (stay tuned...)

Erlang Distribution:

$$f_X(x) = \begin{cases} \frac{a^n x^{n-1} e^{-ax}}{(n-1)!} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Examples: (CDF)

- Probability that the total time of n telephones calls are less than x
- Probability that n atoms will decay within x seconds

Mean: $E(x) = \frac{n}{a}$

Variance: $\sigma^2 = \frac{n}{a^2}$

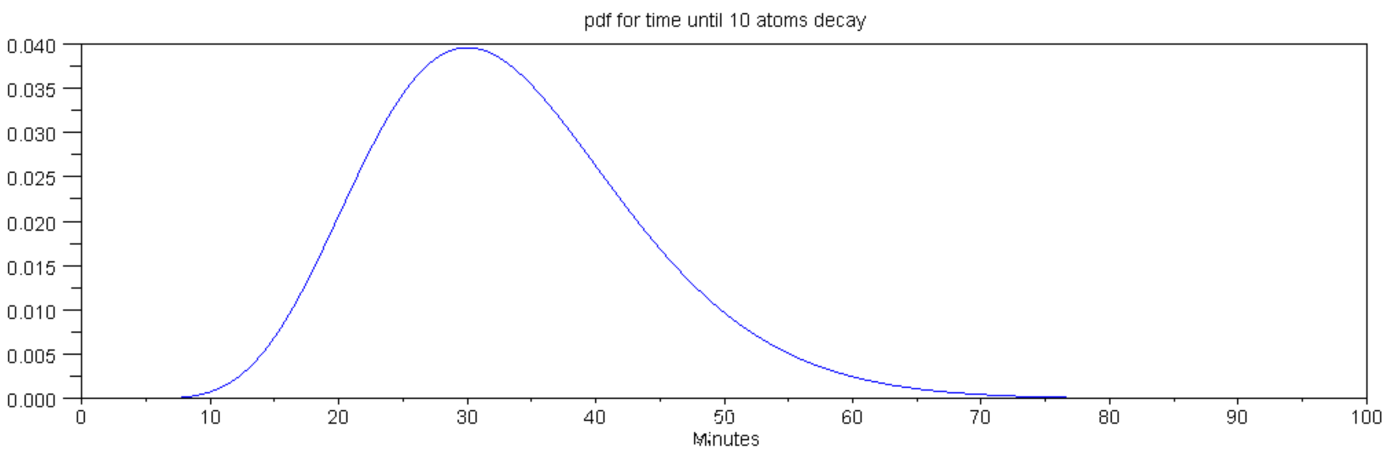
Problem: Plot the pdf and cdf for the the time you have to wait until 10 atoms decay.

In SciLab, the pdf is found from:

```
-->n = 10;
-->a = 0.3;
-->x = [0:0.01:100]';

-->f = (a^n)*(x .^ (n-1)) .* (exp(-a*x)) / factorial(n-1);

-->plot(x,f)
```



The CDF can be found by integrating (the step size used was 0.01 minute):

```
-->F = 0*f;

-->for i=2:length(f)
-->   F(i) = F(i-1) + f(i)*0.01;
-->   end

-->plot(x,F)
```