# Gamma and Poisson Distribution

## ECE 341: Random Processes

### Lecture #14

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

## Gamma and Poisson Distribution

Both are extensions of the exponential distribution:

$$p(t) = ae^{-at}$$

Gamma: Time until k events occur.

- The time until k customers arrive,
- The time until k atoms decay,
- The time until you've been invited to k parties,

Poisson: Probability k events occur in M seconds

- The number of pieces of mail you receive each day (the sending time is exponential)
- The number of cars through in intersection in one minute
- The number of customers arriving at a restaurant in one hour
- Also used to approximate binomial distributions where n is large

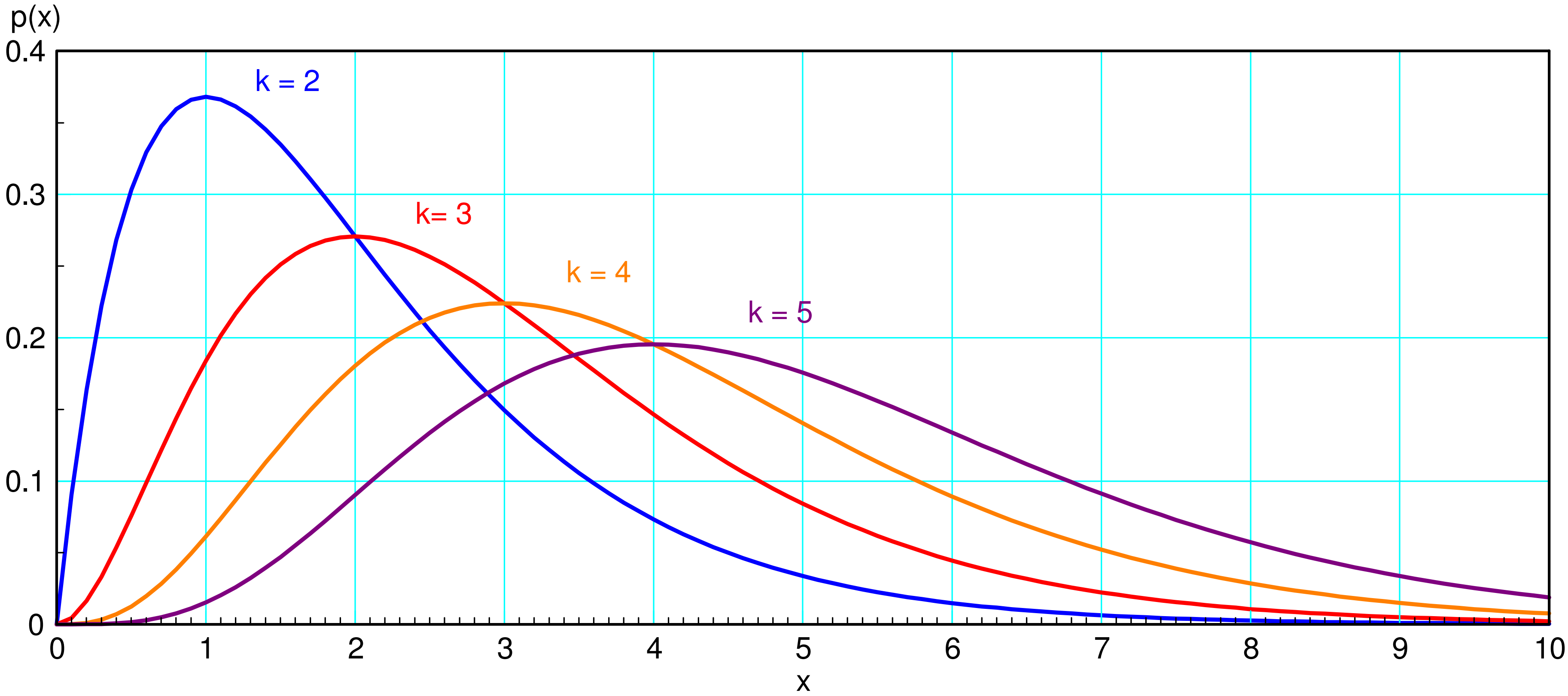
# Gamma Distribution

Exponential: Time until next event

Gamma: Time until k events

| | Exponential | Gamma |
|---|---|---|
| pdf | $f_x = a\ e^{-ax}$ | $f_x = \left(\frac{a^k}{(k-1)!}\right)\ x^{k-1}\ e^{-ax}$ |
| cdf | $F_x = 1 - e^{-ax}$ | complicated |
| mgf | $\left(\frac{a}{s+a}\right)$ | $\left(\frac{a}{s+a}\right)^k$ |
| mean | $\left(\frac{1}{a}\right)$ | $\left(\frac{k}{a}\right)$ |
| variance | $\left(\frac{1}{a^2}\right)$ | $\left(\frac{k}{a^2}\right)$ |

# pdf of a Gamma distribution:




pdf for a Gamma distribution with an average arrival time of 1

## cdf & pdf

- The pdf tells you the probability of x=a
- The cdf tells you the probability that x<a

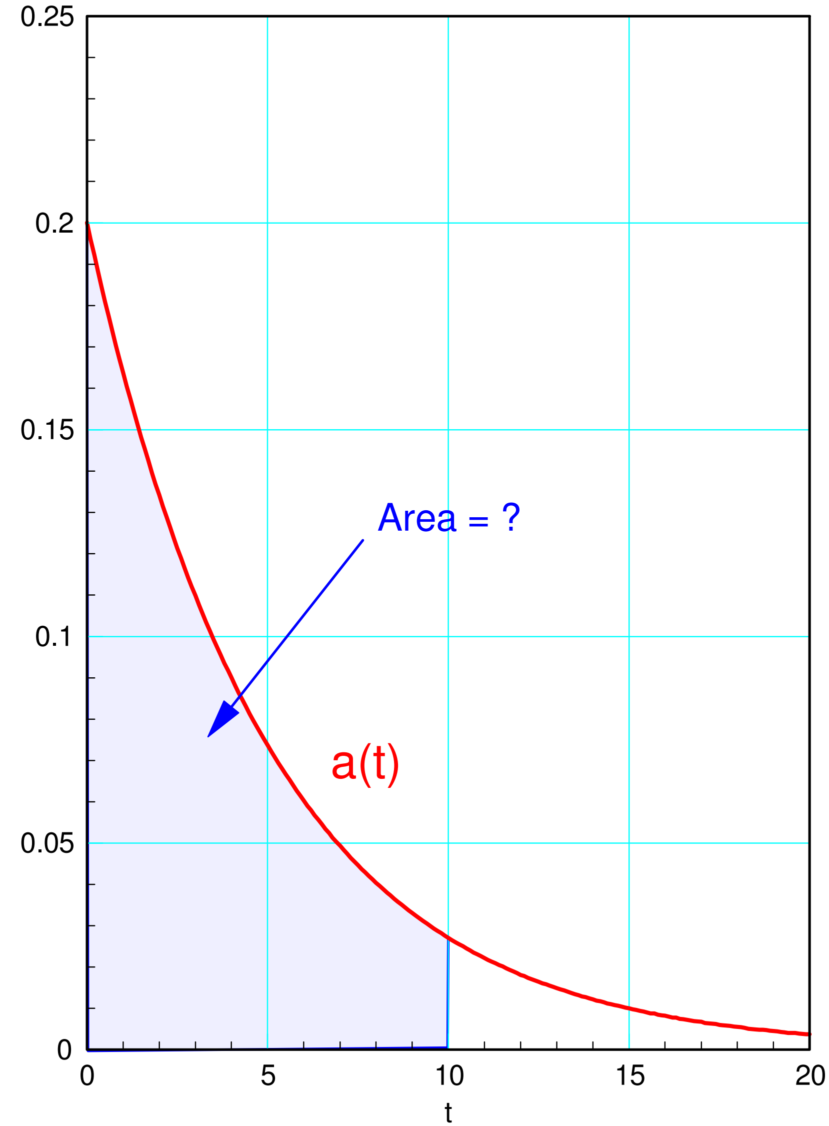
Example: Exponential Distribution

- Assume A has an exponential distribution with a mean of 5 seconds
- Determine the probability that one event occurs in less than 10 seconds

Solution: The pdf is

$$a(t) = \frac{1}{5} \cdot e^{-t/5} \cdot u(t)$$

Find the area to the left of t = 10



The moment generating function is

$$a(s) = \left(\frac{1/5}{s+1/5}\right)$$

Integrate to get the cdf

$$A(s) = \frac{1}{s}a(s) = \left(\frac{1/5}{s+1/5}\right)\left(\frac{1}{s}\right)$$

Take the inverse LaPlace transform

$$A(s) = \left(\frac{a}{s+1/5}\right) + \left(\frac{b}{s}\right)$$

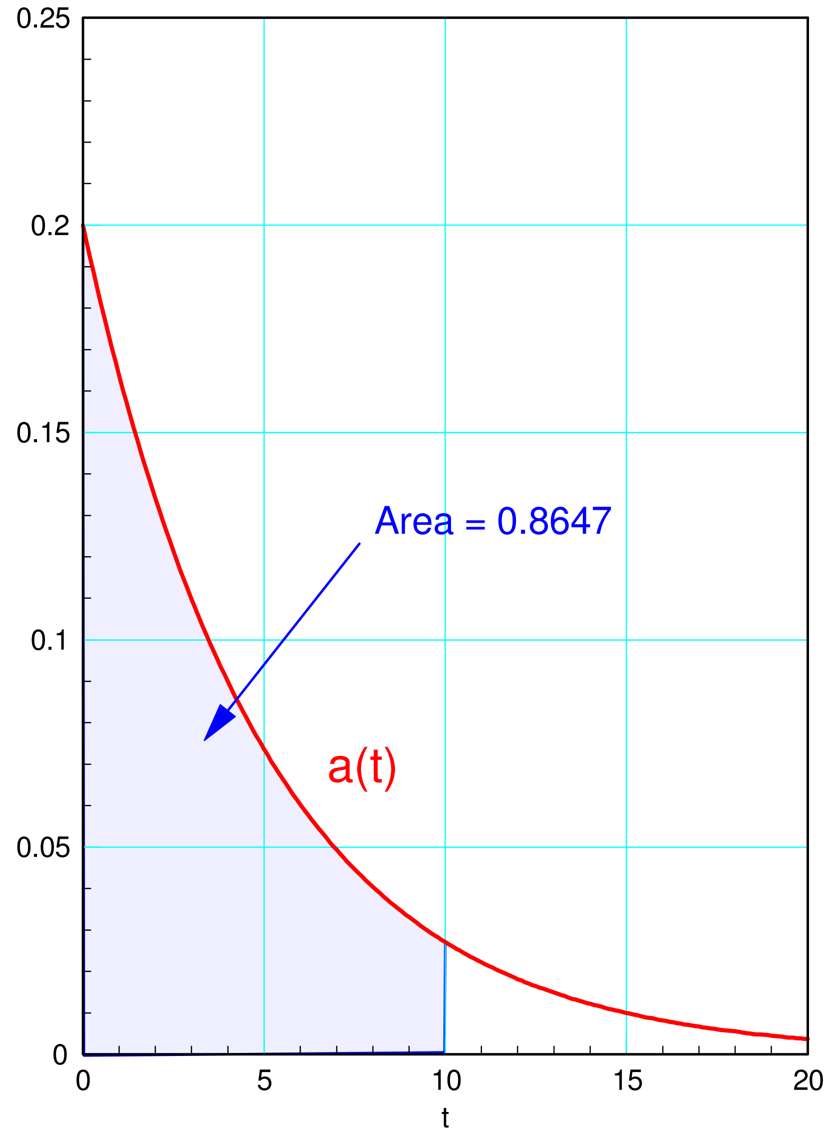
$$A(s) = \left(\frac{1}{s}\right) - \left(\frac{1}{s+1/5}\right)$$

$$cdf(t) = (1 - e^{-t/5})u(t)$$

Plug in t = 10 seconds

$$cdf(10) = 0.8647$$

86.47% chance that one event will happen within 10 seconds



Example 2: Let B be the probability that *three* event happen within 10 seconds

- This is a Gamma distribution with
- k = 3 (time of the 3rd event), and
- a = 1/5 (avg five minutes between events)

Solution: The moment generating function is

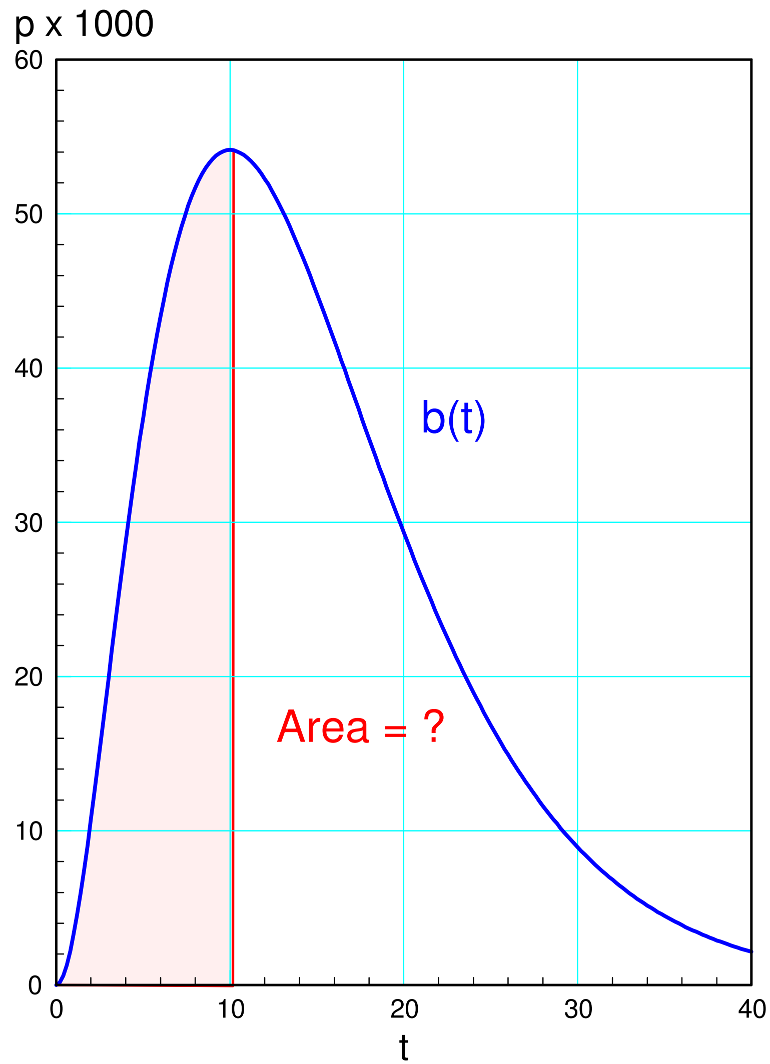
$$\psi(s) = \left(\frac{1/5}{s+1/5}\right)^3$$

From a table of LaPlace transforms

$$\left(\frac{1}{s-a}\right)^3 \rightarrow \frac{1}{2!}\, t^2\, e^{at}\, u(t)$$

Substituting gives you the pdf

$$b(t) = \left(\frac{1}{5}\right)^3 \cdot \frac{1}{2}\, t^2\, e^{-t/5}\, u(t)$$



The cumulative density function (cdf) is the integral of the pdf

$$B(s) = \left(\frac{1/5}{s+1/5}\right)^3 \left(\frac{1}{s}\right)$$

Doing partial fraction expansion

$$B(s) = \left(\frac{a}{(s+1/5)^3}\right) + \left(\frac{b}{(s+1/5)^2}\right) + \left(\frac{c}{s+1/5}\right) + \left(\frac{d}{s}\right)$$

a and d can be found using the cover-up method

$$d = 1, \qquad a = -1/25$$

b and c can be found by placing all terms over a common denominator

$$\left(\frac{1}{5}\right)^3 = as + bs(s+1/5) + cs(s+1/5)^2 + d(s+1/5)^3$$

Match the $s^3$ term

$0 = c + d$ c = -1

Match the $s^2$ term

$0 = b + 2/5c + 3/5d$ b = -1/5

resulting in

$$B(s) = \left(\frac{1/5}{s+1/5}\right)^3 \left(\frac{1}{s}\right)$$

$$B(s) = \left(\frac{-1/25}{(s+1/5)^3}\right) + \left(\frac{-1/5}{(s+1/5)^2}\right) + \left(\frac{-1}{s+1/5}\right) + \left(\frac{1}{s}\right)$$

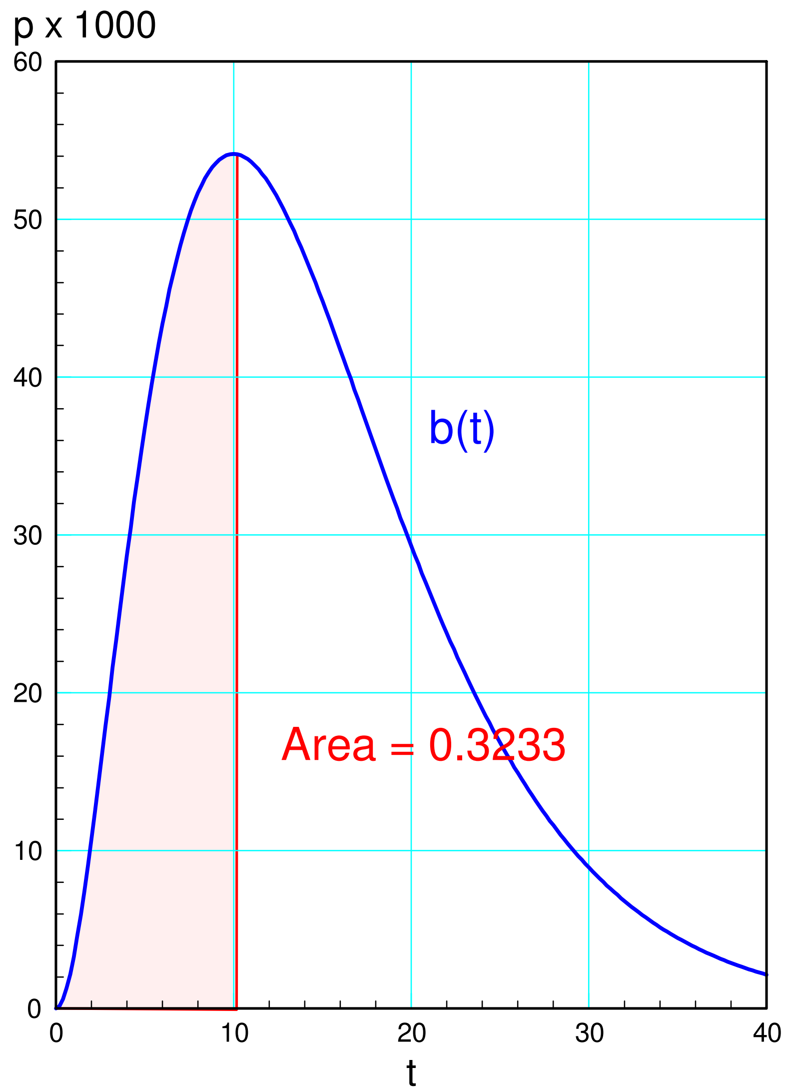
Taking the inverse LaPlace transform gives you the cdf

$$B(t) = \left(\frac{-1}{25} \cdot \frac{1}{2} t^2 e^{-t/5} - \frac{1}{5} t e^{-t/5} - e^{-t/5} + 1\right) u(t)$$

Evaluate at t=10

$$B(t = 10) = 0.3233$$

There is a 32.33% chance that three events will happen in the first 10 minutes



## Gamma Distribution (variation)

- Learning is going on: each time you do a task you get better at it
- Let A be an exponential distribution with a mean of 10 minutes
  - 1st time doing a task
- Let B be an exponential distribution with a mean of 9 minutes
  - 2nd time doing a task
- Let C be an exponential distribution with a mean of 8 minutes
  - 3rd time doing a task

Determine

- The pdf for the time it takes to complete three tasks
- The probability that all three tasks will be done within 30 minutes

## pdf for doing all three tasks:

Use moment-generating functions

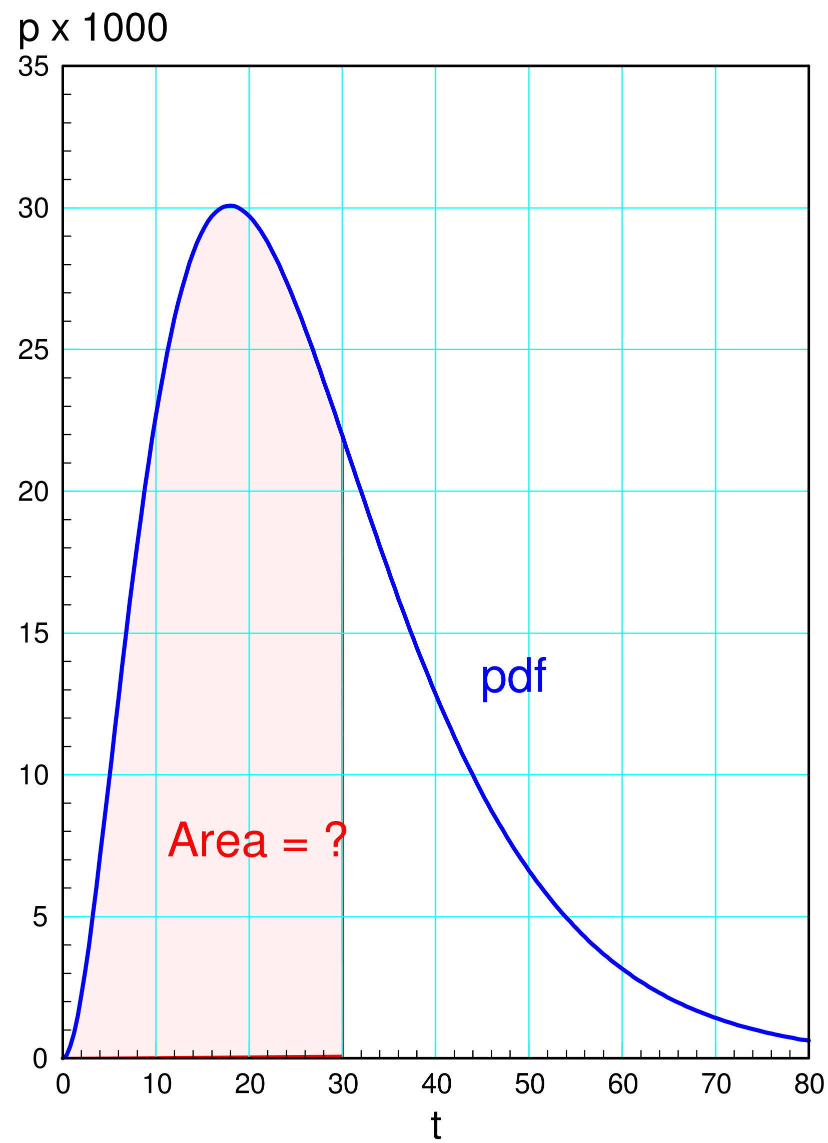
- $A(s) = \left(\frac{1/10}{s+1/10}\right)$
- $B(s) = \left(\frac{1/9}{s+1/9}\right)$
- $C(s) = \left(\frac{1/8}{s+1/8}\right)$
- $y = ABC = \left(\frac{1/10}{s+1/10}\right)\left(\frac{1/9}{s+1/9}\right)\left(\frac{1/8}{s+1/8}\right)$

Do a partial fraction expansion

$$y = \left(\frac{5}{s+1/10}\right) + \left(\frac{-9}{s+1/9}\right) + \left(\frac{4}{s+1/8}\right)$$

The pdf is then

$$y(t) = (5e^{-t/10} - 9e^{-t/9} + 4e^{-t/8})u(t)$$



To find the area left of t=10, use the cdf

$$Y = \frac{1}{s} y(s) = \left(\frac{1/10}{s+1/10}\right)\left(\frac{1/9}{s+1/9}\right)\left(\frac{1/8}{s+1/8}\right)\left(\frac{1}{s}\right)$$

Use partial fraction expansion

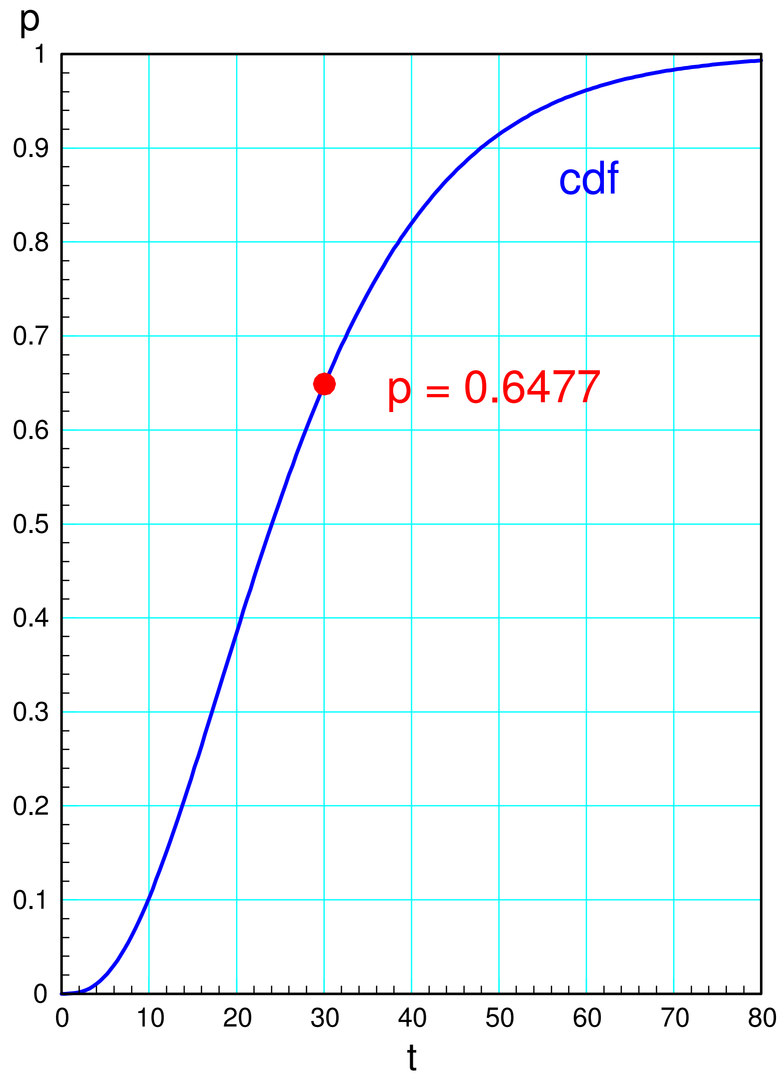
$$Y = \left(\frac{a}{s+1/10}\right) + \left(\frac{b}{s+1/9}\right) + \left(\frac{c}{x+1/8}\right) + \left(\frac{d}{s}\right)$$

$$Y = \left(\frac{-50}{s+1/10}\right) + \left(\frac{81}{s+1/9}\right) + \left(\frac{-32}{x+1/8}\right) + \left(\frac{1}{s}\right)$$

$$cdf(t) = (1 - 50e^{-t/10} + 81e^{-t9} - 32e^{-t/8})u(t)$$

$$cdf(t = 30) = 0.6477$$

*There is a 64.77% chance that all three tasks will be completed in 30 minutes*



## Poisson Distribution.

The Poisson distribution is actually a discrete probability function.

- Gamma: The time until the kth customer arrives, ( Gamma )
- Poisson: The probability that k customers will arrive in a fixed interval

The Poisson distribution is useful if you want to know

- How many cars will go through an intersection in one hour,
- How many customers will arrive in one hour,
- How many patients will go to the emergency room in one day, or
- The number of times your boss will notice you over the course of one week.
- The probability of a binomial distribution when n is large

## Assumptions

(Wikipedia)

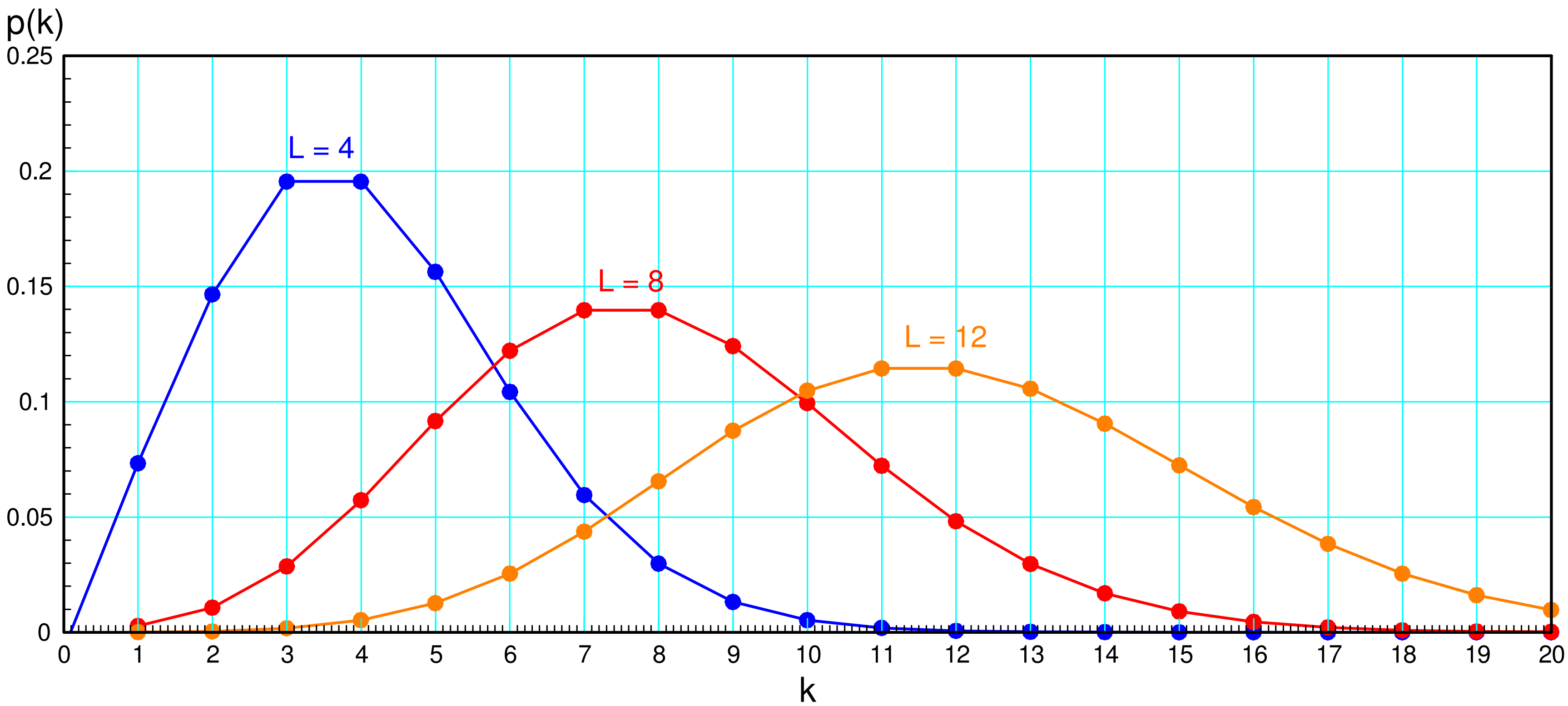
- k is the number of times an event occurs in an interval and k can take values 0, 1, 2, ....
- The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
- The average rate at which events occur is independent of any occurrences. For simplicity, this is usually assumed to be constant, but may in practice vary with time.
- Two events cannot occur at exactly the same instant; instead, at each very small subinterval exactly one event either occurs or does not occur.

The pdf for a Poisson distribution is (wikipedia)

| | Exponential | Poisson |
|---|---|---|
| pdf | $f_x = a\ e^{-ax}$ | $f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ |
| cdf | $F_x = 1 - e^{-ax}$ | complicated |
| mgf | $\left(\frac{a}{s+a}\right)$ | $\exp(\lambda(z-1))$ |
| mean | $\left(\frac{1}{a}\right)$ | $\lambda$ |
| variance | $\left(\frac{1}{a^2}\right)$ | $\lambda$ |

Here, $\lambda = 1/a$ for equating exponential and Poisson processes.

The pdf for a Poisson distribution looks like the following ( $\lambda = 10$ ($a = 1/10$) for illustration purposes).




pdf for a Poisson distribution with $\lambda$ = {4, 8, 12}.
Note that this is a discrete pdf - so only the integer values of k matter

## Poisson Approximation for a Binomial Distribution

A Poisson distribution is also a good approximation for a binomial distribution where the number of rolls is large. For this approximation, match the means:

$$\lambda = np$$

**Example 1:** Plot the probability density function for a binomial distribution with

n = 100, p = 0.05, $\lambda$ = np = 5

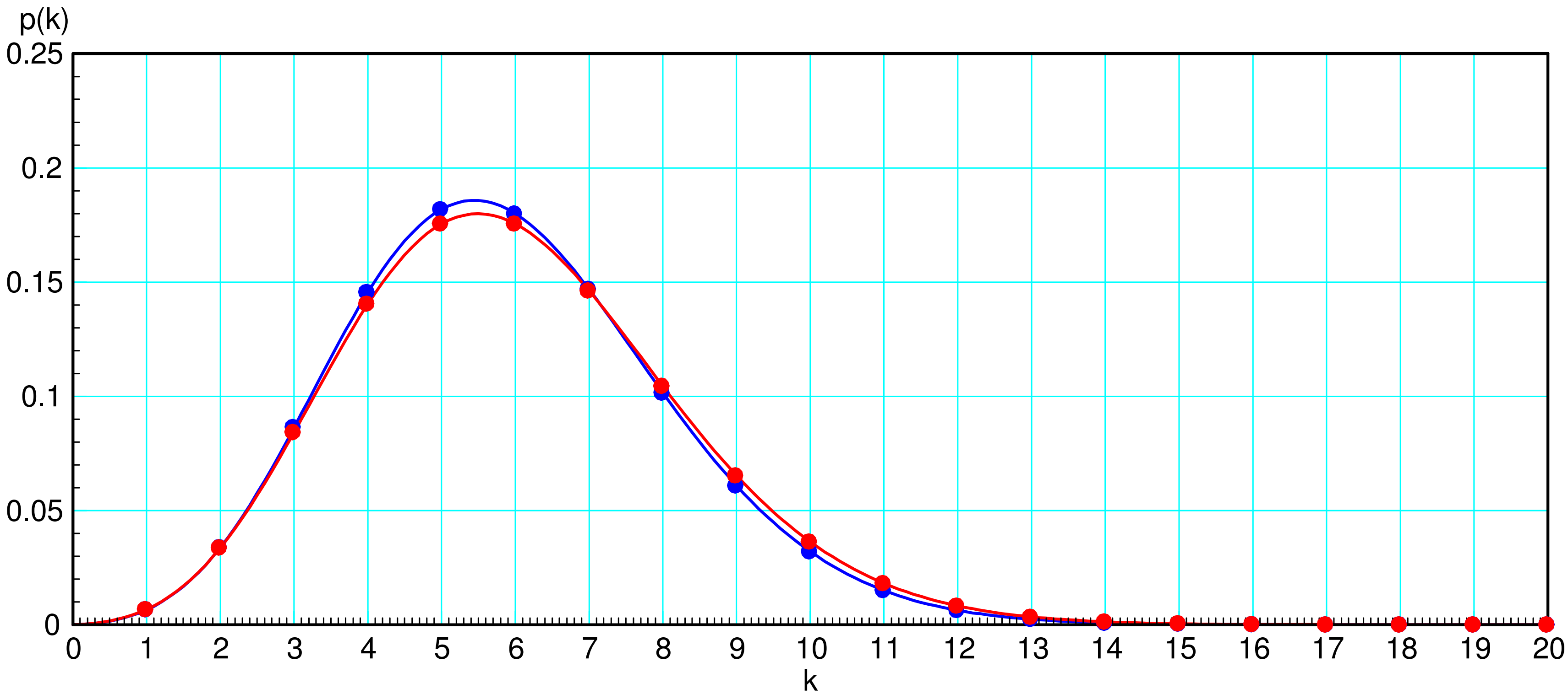
Binomial:

$$f_1(x) = \binom{100}{x}(0.05)^x(0.95)^{100-x}$$

Poisson ( $\lambda = np = 5$ )

$$f_2(x) = \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$

# Poisson vs. Binomial



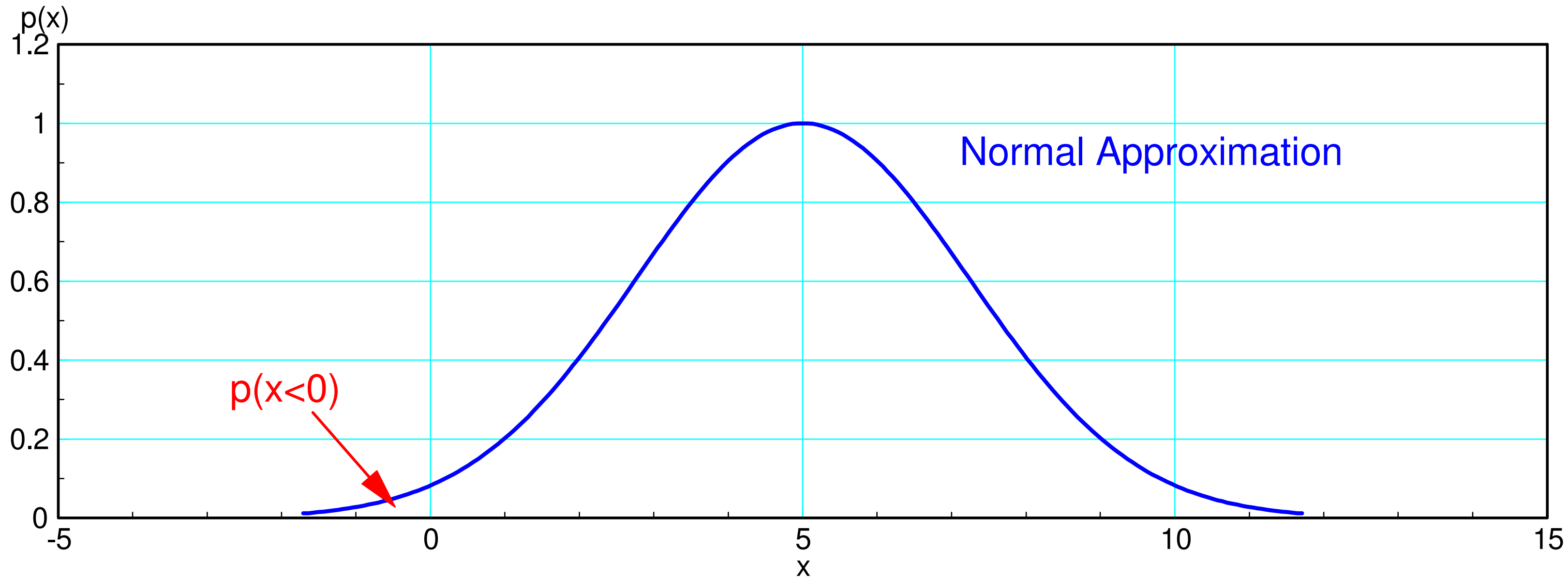

Binomial (blue) vs. Poisson (red) with np = 5

A Poisson approximation is a slightly more complicated approximation for a binomial distribution than a Normal approximation.  It's more accurate however.

- A normal distribution goes from $-\infty$ to $+\infty$
- A Poisson distribution is zero for $k < 0$

In the case of a binomial distribution, you'll never get a negative total.

- A Poisson approximation is slightly more accurate.



**Example 2:** Plot the probability density function for a binomial distribution with

n = 10,000 p = 0.000 5 np = 5

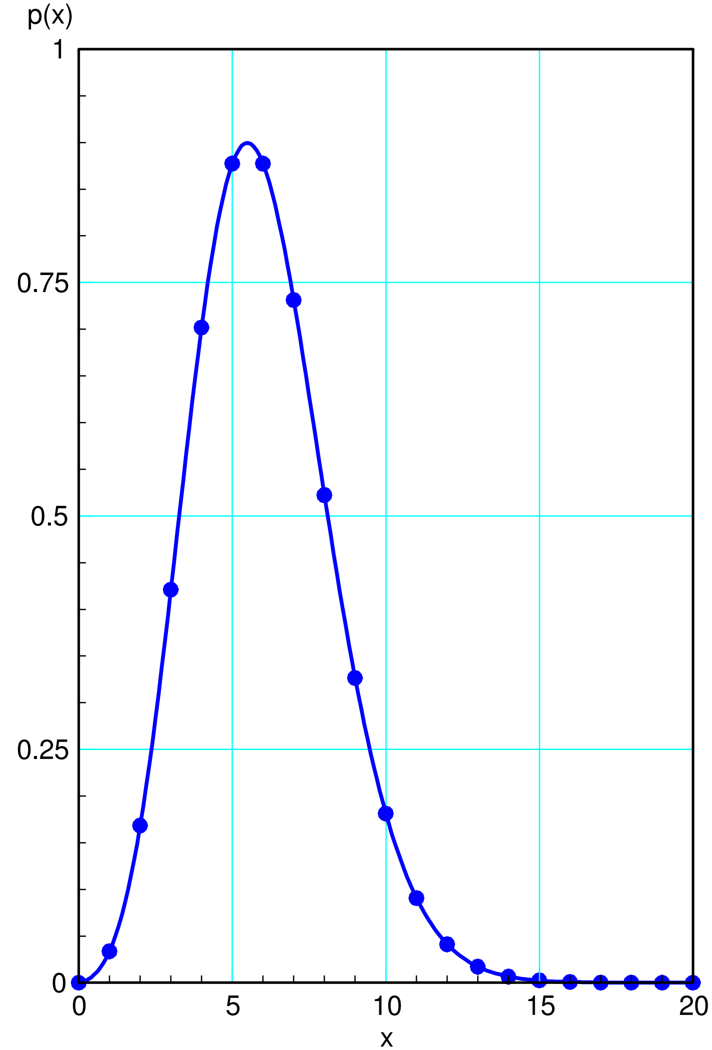
This doesn't work well using a Binomial pdf:

$$f(x) = \binom{10,000}{x}(0.0005)^x(0.9995)^{10,000-x}$$

Not a problem with a Poisson approximation

- np = 5

$$f(x) \approx \frac{1}{x!} \cdot 5^x \cdot e^{-5}$$



## Summary

A Gamma distribution is an exponential distribution

- Where you wait until N events occur
- The moment generating function is $\left(\frac{a}{s+a}\right)^N$

A Poisson distribution

- Is an exponential distribution where you count how many events occur over a time interval
- Is a good approximation for a binomial distribution