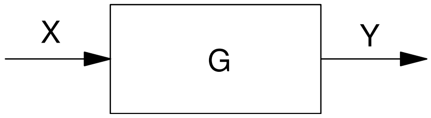
Introduction

Background

Assume you have a system with an input, X, and an output, Y.



The relationship between the input and output can be written mathematically as

$$Y = G \cdot X$$

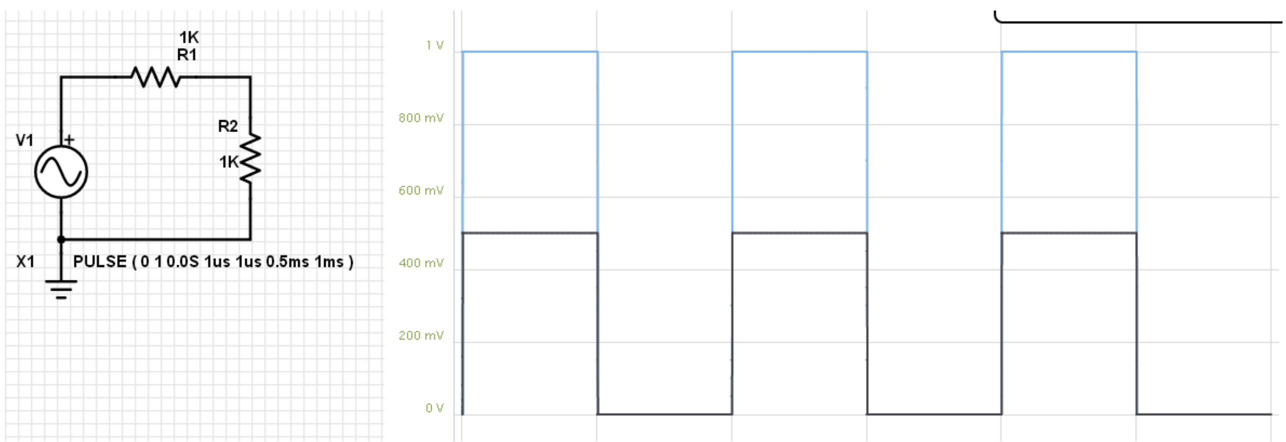
If the input and output are related by a scalar, this is called a *static system.* For example, if you have a voltage divider with R1 = R2 = 1k, the relationship between the input and output is

$$y = \frac{1}{2}x$$

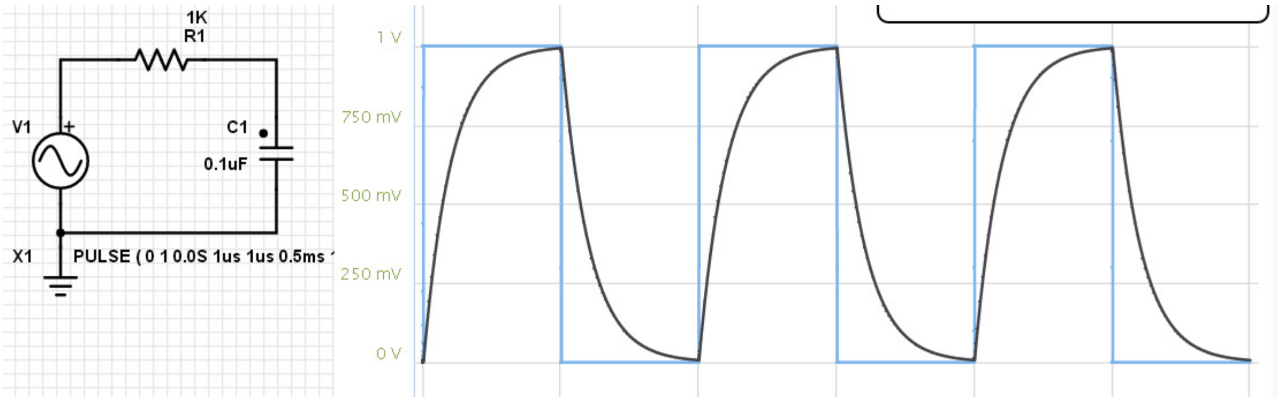
For this circuit

- If x(t) is a sine wave, y(t) is also a sine wave that's 1/2 of the amplitude of the input.
- If x(t) is a square wave, y(t) is also a square wave that's 1/2 of the amplitude of the input.

You can verify this in PartSim:



In contrast, if the circuit contains capacitors or integrators, the relationship between x and y is no longer so simple. For example, if you use an RC filter with a square wave input, the output looks like the following (from PartSim)



This circuit is a *dynamic system*: you need to use a differential equation to describe the relationship between the input and output. This is due to the VI relationship being

$$v = L\frac{di}{dt}$$

for an inductor and

$$i = C\frac{dv}{dt}$$

for a capacitor.

This class, Signals and Systems, looks at how to solve for the output of a dynamic system where the input is known. The tool used to find the output, y(t), depends upon what the input, x(t), looks like. Four different tools are presented herein:

Case 1: Phasor Analysis

$$x(t) = a \cdot \cos(\omega t) \qquad -\infty < t < \infty$$

- x(t) is a sine wave or a sum of sine waves.
- x(t) has been around since $t = -\infty$ meaning you only care about the steady-state solution (initial conditions don't matter).

Case 2: Fourier Transform

$$x(t) = x(t+T) \qquad -\infty < t < \infty$$

- x(t) is periodic in time T.
- x(t) has been around since $t = -\infty$ meaning you only care about the steady-state solution (initial conditions don't matter).

Case 3: LaPlace Transform (zero initial conditions)

$$x(t) = \begin{cases} f(t) & t > 0 \\ 0 & otherwise \end{cases}$$

- x(t) turns on at t=0.
- Initial conditions are zero since x(t) was zero for a long time prior to t=0

LaPlace Transforms (non-zero initial conditions)

$$x(t) = f(t) \qquad t > 0$$

$$y(t = 0) = y_0$$

Most dynamic systems do not have memory: they don't care how you got to t=0. All they care about is what that initial condition is.

Case 4: z-Transform.

$$t = kT$$

where T is the sampling rate (such as 10ms) and k is the sample number. The input and output will then be

$$x(t) = x(kT) = x(k)$$

$$y(t) = y(kT) = y(k)$$

Calculus I and II Review

Suppose you have a system which is described by a differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

Find y(t) assuming

$$x(t) = 5$$

In Calculus, the method you learned was to

- Guess the form of y(t)
- Plug in y(t) into the differential equation
- Solve for the unknown.

Since x(t) is a constant, assume y(t) is in the same form (i.e. a constant)

$$y = a$$

along with

$$\frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} = 0$$

Plugging these into the differential equation results in

$$0 + 0 + 2a = 4 \cdot 5$$

$$a = 10$$

and your solution is

$$y(t) = 10$$

If

$$x(t) = 5e^{-4t}$$

then assume y(t) is in the same form:

$$y(t) = ae^{-4t}$$

This results in

$$\frac{dy}{dt} = -4ae^{-4t}$$

$$\frac{d^2y}{dt^2} = 16ae^{-4t}$$

Substituting back into the differential equations results in

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

$$(16ae^{-4t}) + 3(-4ae^{-4t}) + 2(ae^{-4t}) = 4(5e^{-4t})$$

Canceling the e^{-4t} terms results in

$$16a - 12a + 2a = 20$$

$$a = 2.5$$

and

$$y(t) = 2.5 \cdot e^{-4t}$$

If x(t) is a sine wave

$$x(t) = 4\cos(5t)$$

then assume y(t) is of the form

$$y(t) = a\cos(5t) + b\sin(5t)$$

The derivatives of y(t) are

$$\frac{dy}{dt} = -5a\sin(5t) + 5b\cos(5t)$$

$$\frac{d^2y}{dt^2} = -25a\cos(5t) - 25b\sin(5t)$$

Substituting into the differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 4x$$

$$(-25a\cos(5t) - 25b\sin(5t)) + 3(-5a\sin(5t) + 5b\cos(5t)) + 2(a\cos(5t) + b\sin(5t)) = 4(4\cos(5t))$$

Grouping terms

$$(-25a + 15b + 2a)\cos(5t) + (-25b - 15a + 2b)\sin(5t) = 16\cos(5t) + 0\sin(5t)$$

This gives 2 equations for 2 unknowns

$-23a + 15b = 16$ cos() terms

$-23b - 15a = 0$ sin() terms

Solving

$$a = -0.4880$$

$$b = 0.3183$$

meaning

$$y(t) = -0.4880\cos(5t) + 0.3183\sin(5t)$$

Note that for sinusoidal inputs, you wind up solving 2 equations for 2 unknowns.