# Fourier Transform

## Background:

Suppose you have a filter with an input X and a transfer function G(jw):

$$Y = G(j\omega) \cdot X$$

If x(t) is a sinusoid at frequency $\omega$, y(t) will also be a sinusoid at frequency $\omega$. y(t) is related to x(t) by the gain, gain, G, evaluated at $s = j\omega$.

If x(t) is composed of several sine waves, you can use superposition. The output, y(t) will be the sum of each input times its corresponding gain.

Example: Find y(t) for

$$Y = \left(\frac{20}{(j\omega+2)(j\omega+5)}\right)X$$

$$x(t) = 1 + 2\sin(3t) + 4\sin(5t)$$

Solution: Solve three different problems.

| x(t) | jw | G(s) | y(t) |
|---|---|---|---|
| 1 | jw = 0 | $G(0) = 2$ | 2 |
| $2\sin(3t)$ | jw = j3 | $G(j3) = 0.95\angle-87^0$ | $2 \cdot 0.95\sin(3t - 87^0)$ |
| $4\sin(5t)$ | jw = j5 | $G(j5) = 0.52\angle-113^0$ | $4 \cdot 0.52\sin(5t - 113^0)$ |

y(t) will be the sum of all three terms (by superposition)

$$y(t) = 2 + 1.0\sin(3t - 87^0) + 2.1\sin(5t - 113^0)$$

Note that this only works if the input is composed of sinusoids.

## Fourier Transform

Assume instead that the input is periodic in time T:

$$x(t) = x(t + T)$$

For example, a 10 rad/sec square wave would be

$$x(t) = \begin{cases} 1 & \sin(10t) > 0 \\ 0 & otherwise \end{cases}$$

Since sin(10t) is periodic in $0.2\pi$, x(t) is periodic in $0.2\pi$

$$x(t) = x(t + 0.2\pi)$$

Find y(t). Presently, the tools we have don't work for this problem: x(t) isn't a sine wave.

The solution is typical of engineering solutions:

- Given a difficult problem you can't solve, change the problem to one you can solve.

We know how to solve differential equations when the input is sinusoidal or a sum of sinusoids. Change this problem to a sum of sinusoids.

$$x(t) \approx \sum_i a_i \cos(\omega_i t) + b_i \sin(\omega_i t)$$

Since x(t) is periodic in time T, it is reasonable to assume that all sine and cosine terms will also be periodic in time T. Adding this requirement results in

$$\omega_i = n\omega_0$$

where $\omega_0$ is the fundamental frequency

$$\omega_0 = \frac{2\pi}{T}$$

This results in changing the problem to

$$x(t) \approx \sum_n a_n \cos(n\omega_0 t) + b_n \sin(\omega_0 t)$$

This is termed the *Fourier Series Expansion of x(t)* or Fourier Transform for short.

The Fourier transform is essentially curve fitting. It tries to approximate a periodic function with sinusoids which have the same period. By doing so, you convert a signal which is hard to analyze into a signal composed of sinusoids, which are easy to analyze.

## Converting from the Fourier Series to x(t)

If you have the fourier transform (an, bn terms), finding x(t) is easy: just add up the terms. What going from the Fouier Series to x(t) tells you is:

> ***If you add up a bunch of functions which are periodic in time T, the result will be periodic in time T.***

That deserves a big *duh*. That's pretty obvious.

## Converting from x(t) to the Fourier Series

If you have a function which is periodic in time T, determining the Fourier Series is a bit harder. It's also more significant. What the Fourier Transform tells you is:

> ***If you have a function which is periodic in time T, if that function isn't a pure sine wave, it contains harmonics.***

That's rather significant. It tells you that any periodic waveform which is not a sine wave is composed of a bunch of frequencies and those frequencies are harmonics of the fundamental.

To find the terms for the Fouier series, assume x(t) is periodic:

$$x(t+T) = x(t)$$

and x(t) can be expressed in terms of sine and cosine terms:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o t)$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Note that all sine waves are orthogonal:

$$avg(\cos(\omega_1 t) \cdot \cos(\omega_2 t)) = \begin{cases} \frac{1}{2} & \omega_1 = \omega_2 \\ 0 & otherwise \end{cases}$$

$$avg(\sin(\omega_1 t) \cdot \sin(\omega_2 t)) = \begin{cases} \frac{1}{2} & \omega_1 = \omega_2 \\ 0 & otherwise \end{cases}$$

$$avg(\sin(\omega_1 t) \cdot \cos(\omega_2 t)) = 0$$

This allows you to determine each of the Fourier coefficients as:

$$a_0 = avg(x)$$

$$a_n = 2 \cdot avg(x(t) \cdot \cos(n\omega_0 t))$$

$$b_n = 2 \cdot avg(x(t) \cdot \sin(n\omega_0 t))$$

Note: You can also express x(t) in polar form

$$x(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_o t + \theta_n)$$

where

$$a_n - jb_n = c_n \angle \theta_n$$

or complex exponential form:

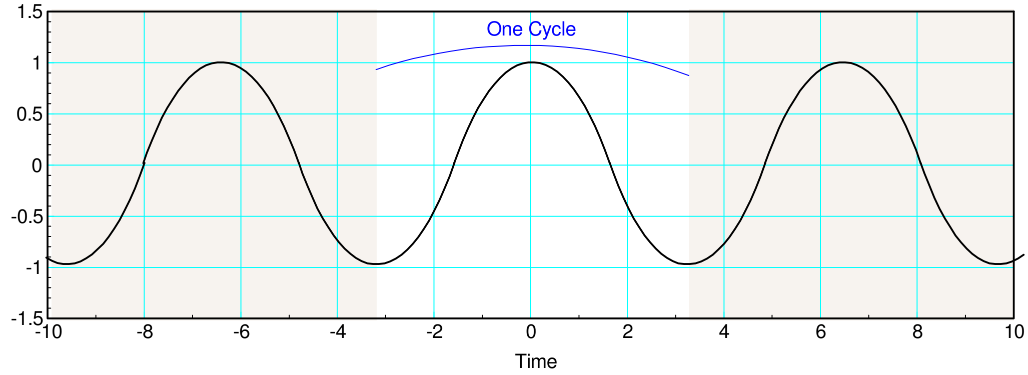
$$x(t) = a_0 + \sum_{n=1}^{\infty} c_n e^{j(n\omega_o t + \theta_n)}$$

All three forms are equivalent - it's just what you're personal preference is. I personally like the first form.

## Example 1: Sine Wave

Find the Fourier trasnform for a 1 rad/sec cosine wave

$$x(t) = \cos(t)$$




Solution: DC term

$$a_0 = \frac{1}{T}\int_T x(t) \cdot dt$$

$$a_0 = 0$$

Cosine terms

$$a_n = \frac{2}{T}\int_T x(t) \cdot \cos(nt) \cdot dt$$

$$a_n = \frac{2}{T}\int_{-\pi}^{\pi} \cos(t) \cdot \cos(nt) \cdot dt$$

$$a_n = \begin{cases} \frac{2}{T}\int_{-\pi}^{\pi} \cos^2(t) \cdot dt & n = 1 \\ 0 & otherwise \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right) \cdot dt$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2}\right) \cdot dt$$

$$a_1 = 1$$

The Fourier transform for

$$x(t) = \cos(t)$$

is

$$x(t) = \cos(t)$$

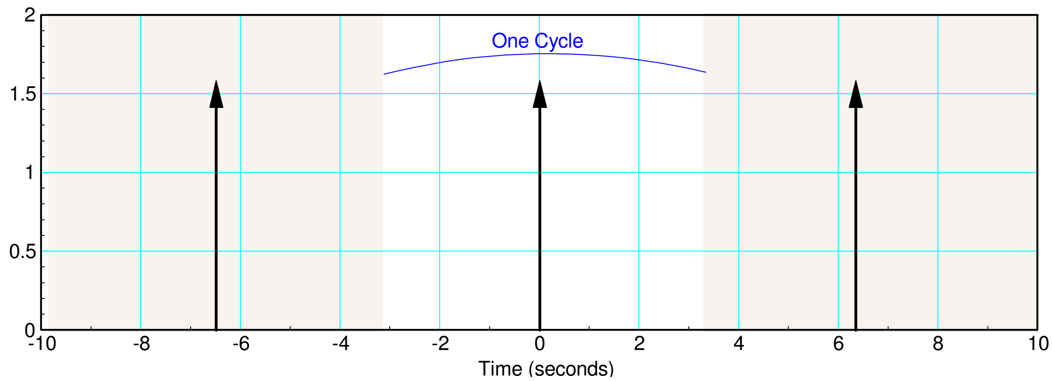
If the function is already expressed in terms of sine and cosine functions, it's already in the form of its Fourier transform.

## Example #2: Delta Function

Let x(t) be a 1 rad/sec delta function

$$x(t) = x(t+2\pi)$$

$$x(t) = \delta(t)$$




Find the Fourier transform: X(jw)

Solution: The DC term is

$$a_0 = \frac{1}{T}\int_T x(t)\cdot dt$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t)\cdot dt$$

$$a_0 = \frac{1}{2\pi}$$

The cosine terms are (note: $\omega_0 = \frac{2\pi}{T} = 1$ )

$$a_n = \frac{2}{T}\int_T x(t)\cdot\cos(nt)\cdot dt$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \delta(t)\cdot\cos(nt)\cdot dt$$

$$a_n = \frac{1}{\pi}$$

The sine terms are

$$b_n = \frac{2}{T}\int_T x(t)\cdot\sin(nt)\cdot dt$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \delta(t)\cdot\sin(nt)\cdot dt$$

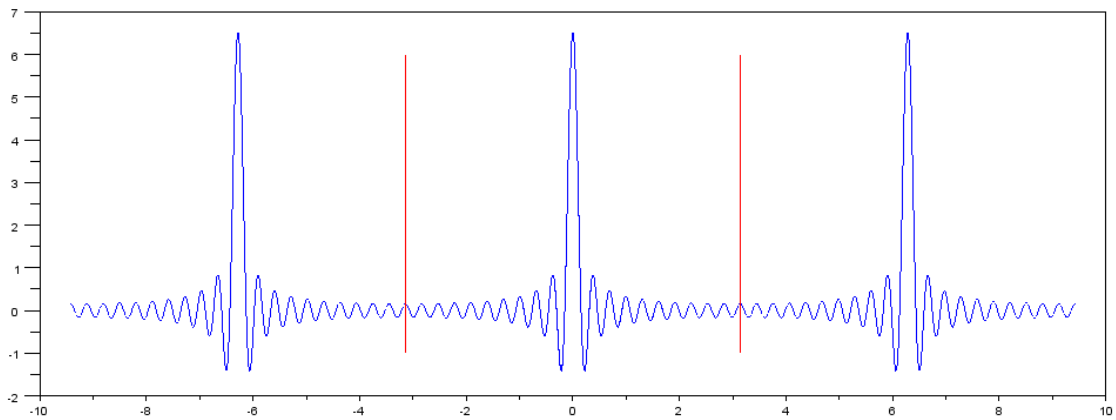
$$b_n = 0$$

So,

$$\delta(t) = \frac{1}{2\pi} + \sum_{n=1..\infty} \frac{1}{\pi}\cos(n\pi t)$$

You can check this in Matlab by plotting this out to 20 harmonics (should go to infinity). Just for kicks, plot x(t) from -3pi to +3pi

```
-->t = [-3*pi:0.001:3*pi]';
-->x = 0*t + 1/(2*pi);

-->for n=1:20
-->   x = x + cos(n*t)/pi;
-->   end

-->plot(t,x)
```




Fourier Series Approximation for a delta function taken out to 20 harmonics

The Fourier coefficients can also be shown in a table format:

| harmonic | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| an | 0.1592 | 0.3183 | 0.3183 | 0.3183 | 0.3183 | 0.3183 | 0.3183 | 0.3183 |
| bn | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

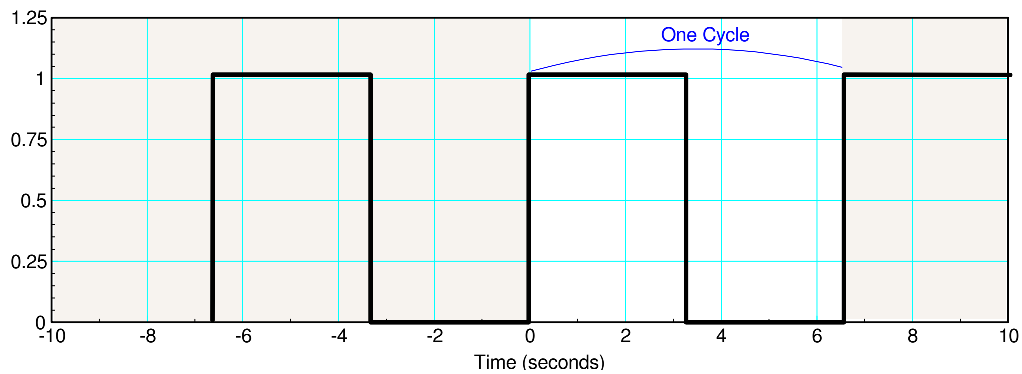
Note that a delta function contains an infinite number of harmonics, each with the same amplitude.

## Example 2: Square Wave

Find the Fourier Transform for a 1 rad/sec 50% Duty Cycle Square Wave.

$$x(t) = x(t+2\pi)$$

$$x(t) = \begin{cases} 1 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$




Solution: The DC term is

$$a_0 = \frac{1}{T}\int_T x(t) \cdot dt = \frac{1}{2}$$

The cosine terms

$$a_n = \frac{2}{T}\int_T x(t) \cdot \cos(nt) \cdot dt$$

$$a_n = \frac{2}{2\pi}\int_0^\pi 1 \cdot \cos(nt) \cdot dt$$

$$a_n = \frac{1}{\pi} \cdot \left(\frac{1}{n}\sin(nt)\right)_0^\pi$$

$$a_n = 0$$

The sine terms

$$b_n = \frac{2}{T}\int_T x(t) \cdot \sin(nt) \cdot dt$$

$$b_n = \frac{2}{2\pi}\int_0^\pi 1 \cdot \sin(nt) \cdot dt$$

$$b_n = \frac{1}{\pi} \cdot \left(\frac{-1}{n}\cos(nt)\right)_0^\pi$$

$$b_n = \frac{1}{\pi} \cdot \left(\frac{1+(-1)^n}{n}\right)$$

$$b_n = \begin{cases} \frac{2}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

So, the Fourier Transform for a 0V - 1V square wave is

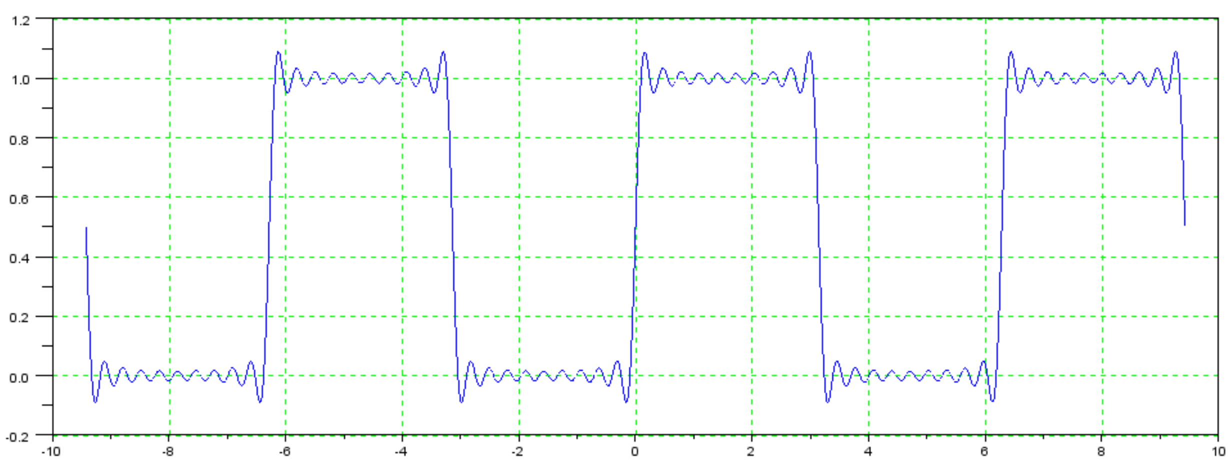
$$x(t) = \frac{1}{2} + \sum_{n=1,3,5,...} \frac{2}{n\pi} \sin(nt)$$

A table of the Fourier coefficients is

| harmonic | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| an | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| bn | 0 | 0.6366 | 0 | 0.2122 | 0 | 0.1273 | 0 | 0.0909 |

In Matlab, plotting x(t) out to its 20th harmonic results in the following:

```
x = 0.5 + 0*t;

for i=1:10
   n = 2*i-1;
   x = x + 2/(n*pi) * sin(n*t);
   end

plot(t,x)
```

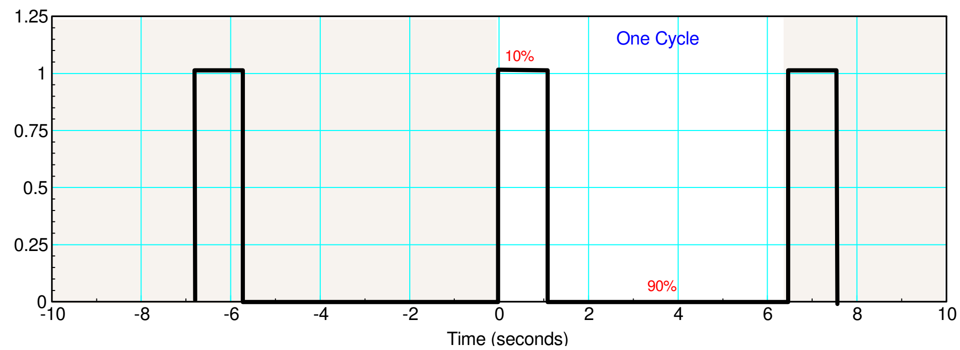



Again, it isn't a perfect square wave. To get that, you'd have to go out to infinity.

## Example 3: A 10% Duty Cycle Square Wave

$$x(t) = x(t + 2\pi)$$

$$x(t) = \begin{cases} 1 & 0 < t < \frac{2\pi}{10} \\ 0 & \frac{2\pi}{10} < t < 2\pi \end{cases}$$




The DC term is:

$$a_0 = \frac{1}{T} \int_T x(t) \cdot dt = \frac{1}{10}$$

The cosine terms:

$$a_n = \frac{2}{T} \int_T x(t) \cdot \cos(nt) \cdot dt$$

$$a_n = \frac{2}{2\pi} \int_0^{\pi/5} 1 \cdot \cos(nt) \cdot dt$$

$$a_n = \frac{1}{\pi} \left( \frac{1}{n} \sin(nt) \right)_0^{\pi/5}$$

$$a_n = \frac{1}{n\pi} \sin\left(\frac{n\pi}{5}\right)$$

The sine terms:

$$b_n = \frac{2}{T} \int_T x(t) \cdot \sin(nt) \cdot dt$$

$$b_n = \frac{2}{2\pi} \int_0^{\pi/5} 1 \cdot \sin(nt) \cdot dt$$

$$b_n = \frac{1}{\pi} \left( \frac{-1}{n} \cos(nt) \right)_0^{\pi/5}$$

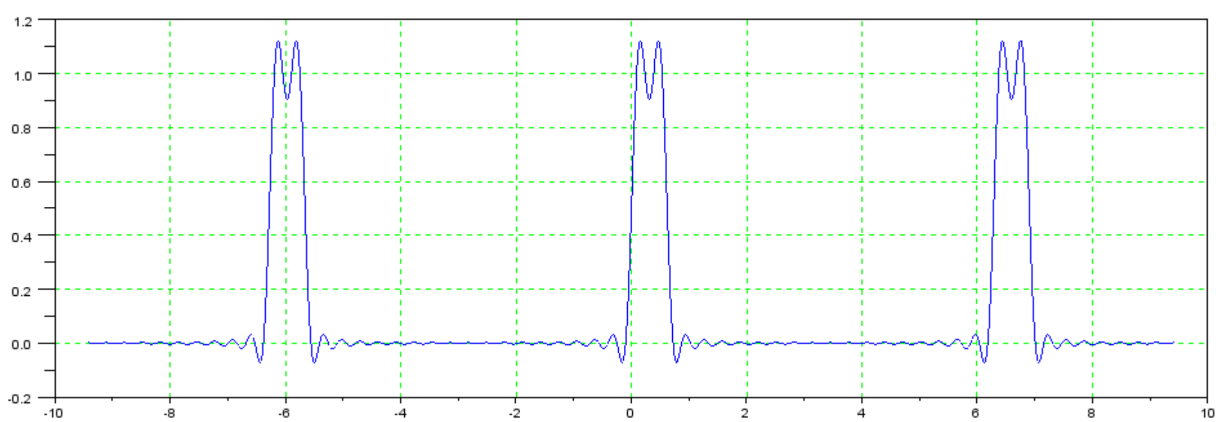
$$b_n = \frac{1}{n\pi} \left( 1 - \cos\left(\frac{n\pi}{5}\right) \right)$$

A table of Fourier coefficients is:

| harmonic | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| an | 0.1 | 0.0013 | 0.0043 | 0.0064 | 0.0053 | 0 | -0.0080 | -0.0151 |
| bn | 0 | 0.0004 | 0.0031 | 0.0089 | 0.0164 | 0.0227 | 0.0246 | 0.0280 |

Adding up the first 20 terms in Matlab

```
an = zeros(20,1);
bn = zeros(20,1);

n = [1:20]';

an = (1 ./ (n*pi)) .* sin( n*pi/5)
bn = (1 ./ (n*pi)) .* ( 1 - cos( n*pi/5 ) );

x = 0.1 + 0*t;

for n=1:20
   x = x + an(n) * cos(n*t)  +  bn(n) * sin(n*t);
   end

plot(t,x)
```




Fourier Series approximation to a 10% duty cycle square wave, taken out to the 20th harmonic