# Properties of LaPlace Transforms

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Let's start with some properties of LaPlace transforms:

Linearity: $af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$

Convolution: $f(t) * *g(t) \Leftrightarrow F(s) \cdot G(s)$

Differentiation: $\frac{dy}{dt} \Leftrightarrow sY - y(0)$

$$\frac{d^2y}{dt^2} \Leftrightarrow s^2Y - sy(0) - \frac{dy(0)}{dt}$$

Integration: $\int_0^t x(\tau)d\tau = \frac{1}{s}X(s)$

Delay $x(t-T) \Leftrightarrow e^{-sT}X(s)$

## Proofs:

**Linearity**:

$$\begin{aligned} L(af(t) + bg(t)) &= \int_{-\infty}^{\infty} (af(t) + bg(t)) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt \\ &= a\int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \\ &= aF(s) + bG(s) \end{aligned}$$

**Convolution:**

$$f(t) * *g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau$$

$$L(f(t) * *g(t)) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau$$

$$= \left( \int_{-\infty}^{\infty} f(t-\tau) \cdot e^{-st} \cdot dt \right) \cdot \left( \int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

**Differentiation:**

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for t<0)

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

$$= (x \cdot e^{-st})_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt$$

$$= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= sX - x(0)$$

**Integration:**

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

Integrate by parts.

$$\int a\cdot b'\cdot dt = ab - \int a'\cdot b\cdot dt$$

Let

$$a = \int_0^t x(\tau)\cdot d\tau$$

$$b' = e^{-st}$$

then

$$a' = x$$

$$b = \tfrac{-1}{s}e^{-st}$$

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

$$= \left(\int_0^t x(\tau)\cdot d\tau\cdot\tfrac{-1}{s}e^{-st}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x\cdot\tfrac{-1}{s}e^{-st}\cdot dt$$

Assuming the function vanishes at infinity

$$= \tfrac{1}{s}\int_{-\infty}^{\infty} x\cdot dt$$

$$= \left(\tfrac{1}{s}\right)X(s)$$

**Time Delay**

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(t-T)\cdot e^{-st}\cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(\tau)\cdot e^{-s(\tau+T)}\cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\cdot e^{-s\tau}\cdot e^{-sT}\cdot d\tau$$

$$= e^{-sT}\cdot\int_{-\infty}^{\infty} x(\tau)\cdot e^{-s\tau}\cdot d\tau$$

$$= e^{-sT}\cdot X(s)$$

## Using Properties of LaPlace Transform

**Unit Step:** Find the LaPlace transform of

$$x(t) = u(t)$$

Write this as

$$x(t) = \int \delta(t) \cdot dt$$

Use the integration property

$$X(s) = \left(\frac{1}{s}\right) \cdot 1$$

**Unit Ramp:** Find the LaPlace transform of

$$x(t) = t \cdot u(t)$$

Rewrite this as

$$x(t) = \int u(t) \cdot dt$$

$$X(s) = \left(\frac{1}{s}\right) \cdot \left(\frac{1}{s}\right) = \frac{1}{s^2}$$

**Unit Parabola:** Find the LaPlace transform of

$$x(t) = t^2 \cdot u(t)$$

Rewrite this as

$$x(t) = \int 2t \cdot u(t) \cdot dt$$

$$X(s) = \left(\frac{1}{s}\right)\left(\frac{2}{s^2}\right) = \left(\frac{2}{s^3}\right)$$

**Unit Pulse:**

$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & otherwise \end{cases}$$

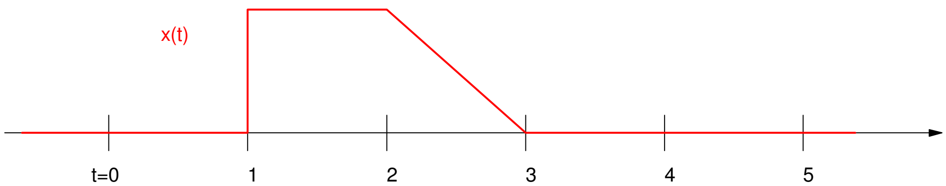
Rewrite this as

$$x(t) = u(t) - u(t-1)$$

$$X(s) = \left(\frac{1}{s}\right) - (e^{-s})\left(\frac{1}{s}\right)$$

$$X(s) = \left(\frac{1-e^{-s}}{s}\right)$$

You can also find the LaPlace transform for other functions. For example, find the LaPlace transform for x(t):




Painful Solution: Express x(t). Define two windows in the intervals (1,2) and (2,3). Use u() to turn on different functions in each window.

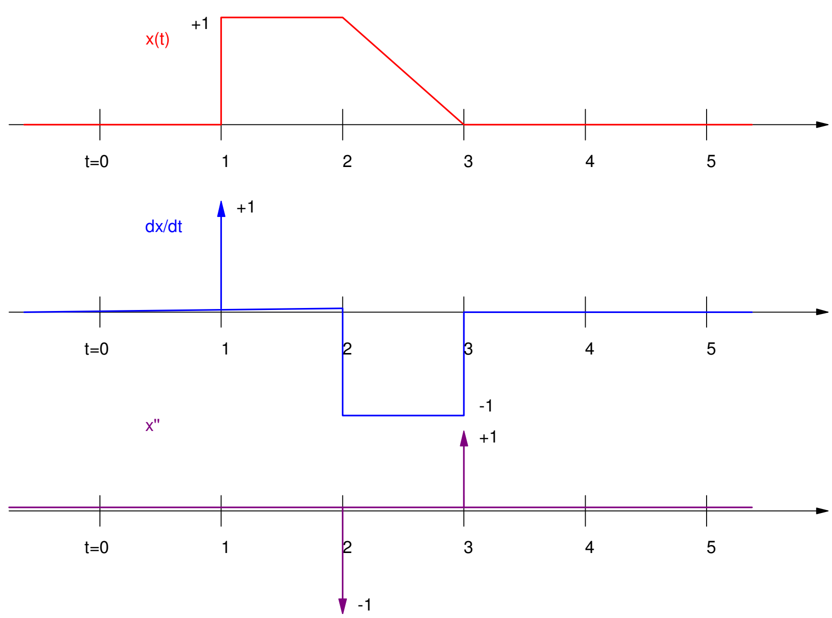
$$x(t) = 1 \cdot (u(t-1) - u(t-2)) + (3-t) \cdot (u(t-2) - u(t-3))$$

Now plug into the definition

$$X(s) = \int_{0^-}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

After about an hour, you'll have X(s).

Easier Solution: Differentiate until you get delta funcitons. Delta functions are EASY to convert to LaPlace:




$$X(s) = 0$$

(no delta functions)

$$+\left(\frac{1}{s}\right)e^{-s}$$

(add in the delta function at t=1, integrated one time to get back to x(t)

$$+\left(\frac{1}{s}\right)^2(-e^{-2s} + e^{-3s})$$

(add in two more delta funcitons, integrated twice to get to x)

The net result is

$$\boxed{X(s) = \left(\frac{1}{s^2}\right)(-e^{-2s} + e^{-3s}) + \left(\frac{1}{s}\right)(e^{-s})}$$