z-Transform

Table of z-transforms:

$$x(k) \leftrightarrow X(z)$$

$$\delta(k) \leftrightarrow 1$$

$$u(k) \leftrightarrow \left(\frac{z}{z-1}\right)$$

$$ku(k) \leftrightarrow \left(\frac{z}{(z-1)^2}\right)$$

$$a^k u(k) \leftrightarrow \left(\frac{z}{z-a}\right)$$

$$2ba^k \cos(k\theta + \phi)u(k) \leftrightarrow (b\angle\phi)\left(\frac{z}{z-a\angle\theta}\right) + (b\angle-\phi)\left(\frac{z}{z-a\angle-\theta}\right)$$

$$\cos(k\theta)u(k) \leftrightarrow \left(\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right)$$

$$\sin(k\theta)u(k) \leftrightarrow \left(\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right)$$

Introduction

LaPlace transform convert *differential equations* into algebraic equations in 's' - with the assumption that algebra is easier than calculus. The basic assumption behind the LaPlace transform is that all functions are in the form of

$$y(t) = a \cdot e^{st}$$

With this assumption, differentiation becomes multiplication by 's'

$$\frac{dy}{dt} = s \cdot ae^{st} = sy$$

Likewise, you can read 'sy' as 'the derivative of y'.

The z-transform converts *difference equations* into algebraic equations in 'z'. The basic assumption behind z-transform is that all functions are in the form of

$$y(k) = a \cdot z^k$$

With this assumption, moving forward in time becomes multiplying by 'z'

$$y(k+1) = a \cdot z^{k+1} = z \cdot az^k = zy(k)$$

Likewise, you can read 'zy' as 'the next value of y'.

LaPlace transforms also allow you to solve differential equations with forcing functions without having to use colvolution. Similarly, z-transforms allow you to solve difference equations with forcing functions without having to use convolution as well.

z-transform

The definition of the z-transform is

$$X(z) = \sum_{n=1}^{\infty} x(n) z^{-n}$$

Let's start with the z-transform of a delta function. In discrete-time, a delta function is

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$$

The z-transform for a delta function is likewise 1

$$\delta(k) \leftrightarrow 1$$

Almost by definition, if you have x(k):

k	-3	-2	-1	0	1	2	3	4	5
x(k)	0	0	0	x(0)	x(1)	x(2)	x(3)	x(4)	x(5)

this can be written as

$$x(k) = x(0)\delta(k) + x(1)\delta(k-1) + x(2)\delta(k-2) + \ldots$$

Since 'zx' means 'the next value of x'

$$zx(k) = x(k+1)$$

you also have 'z-1x' means 'the previous value of x' or 'x delayed one sample'

$$z^{-1}x(k) = x(k-1)$$

Similarly, you can write x(k) as

$$x = x(0) + z^{-1}x(1) + z^{-2}x(2) + z^{-3}x(3) + \ldots$$

which is the definition of the z-transform.

z-Transforms for Different Functions, x(k)

Delta Function: As shown previously, the z-transform for a delta function is one

$$\delta(k) \leftrightarrow 1$$

It's value vs. time is

k	-3	-2	-1	0	1	2	3	4	5
$\delta(k)$	0	0	0	1	0	0	0	0	0

Unit Step: The unit step function is

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & otherwise \end{cases}$$

To find the z-transform, use the following trick.

- Start with u(k)
- Subtract u(k-1)
- This gives a delta function. Take the z-transform since we know the z-transform of a delta function
- The result must be the z-transform of u(k)

k	-3	-2	-1	0	1	2	3	4	5
$u(k)$	0	0	0	1	1	1	1	1	1
$z^{-1}u(k)$	0	0	0	0	1	1	1	1	1
$= (1-z^{-1})u(k)$	0	0	0	1	0	0	0	0	0

The difference is a delta function, which has a LaPlace transform of 1, so

$$(1-z^{-1})u(k) \leftrightarrow 1$$

$$u(k) \leftrightarrow \left(\frac{1}{1-z^{-1}}\right) = \left(\frac{z}{z-1}\right)$$

$$u(k) \leftrightarrow \left(\frac{z}{z-1}\right)$$

Unit Ramp:

$$x(k) = ku(k)$$

Use the trick of finding what to subtract to give something you know: a unit step or a delta function

k	-3	-2	-1	0	1	2	3	4	5
$ku(k)$	0	0	0	0	1	2	3	4	5
$z \cdot ku(k)$	0	0	0	1	2	3	4	5	6
$= (1-z)ku(k)$	0	0	0	-1	-1	-1	-1	-1	-1

We know the z-transform of the result: a negative step. So

$$(1-z)ku(k) \leftrightarrow -\left(\frac{z}{z-1}\right)$$

$$ku(k) \leftrightarrow \left(\frac{z}{(z-1)^2}\right)$$

Decaying Exponential:

$$x(k) = a^k u(k)$$

Again, try to find what it takes to result in a step or a delta function: Start with delaying x(k)

k	-3	-2	-1	0	1	2	3	4	5
$a^k u(k)$	0	0	0	1	a	a^2	a^3	a^4	a^5
$z^{-1} \cdot a^k u(k)$	0	0	0	0	1	a	a^2	a^3	a^4

This doesn't quite work. Multiply the delayed signal by 'a' and subtract

k	-3	-2	-1	0	1	2	3	4	5
$a^k u(k)$	0	0	0	1	a	a^2	a^3	a^4	a^5
$az^{-1} \cdot a^k u(k)$	0	0	0	0	a	a^2	a^3	a^4	a^5
$(1-az^{-1}) \cdot a^k u(k)$	0	0	0	1	0	0	0	0	0

This is a delta function - whose z-transform we know

$$(1-az^{-1}) \cdot a^k u(k) \leftrightarrow 1$$

$$a^k u(k) \leftrightarrow \left(\frac{1}{1-az^{-1}}\right)$$

$$a^k u(k) \leftrightarrow \left(\frac{z}{z-a}\right)$$

Exponential Sinusoid:

If 'a' is a complex number in the previous case, you have a decaying sinusoid. Let

$$X(z) = (b\angle\phi)\left(\frac{z}{z-a\angle\theta}\right) + (b\angle-\phi)\left(\frac{z}{z-a\angle-\theta}\right)$$

From the previous solution

$$x(k) = (b\angle\phi)\cdot(a\angle\theta)^k + (b\angle-\phi)\cdot(a\angle-\theta)^k$$

This is shorthand notation for

$$x(k) = (be^{j\phi})\cdot(ae^{j\theta})^k + (be^{-j\phi})\cdot(ae^{-j\theta})^k$$

$$= (be^{j\phi})\cdot(a^k e^{jk\theta}) + (be^{-j\phi})\cdot(a^k e^{-jk\theta})$$

$$= ba^k \cdot e^{j(k\theta+\phi)} + ba^k \cdot e^{-j(k\theta+\phi)}$$

$$= 2ba^k \cdot \left(\frac{e^{j(k\theta+\phi)}+e^{-j(k\theta+\phi)}}{2}\right)$$

$$= 2ba^k \cos(k\theta+\phi)$$

$$2ba^k \cos(k\theta+\phi) \leftrightarrow (b\angle\phi)\left(\frac{z}{z-a\angle\theta}\right) + (b\angle-\phi)\left(\frac{z}{z-a\angle-\theta}\right)$$

With this, you can find the z-transform and inverse-z transform of most functions.

z-Transform Examples

Example 1: Find the z-transform for

$$x(k) = (3 + 4\cdot 0.9^k + 5\cdot 0.8^k)u(k)$$

Solution: Treat as three separate inputs

$$X(z) = 3\left(\frac{z}{z-1}\right) + 4\left(\frac{z}{z-0.9}\right) + 5\left(\frac{z}{z-0.8}\right)$$

(You could simplify if you like - but this answer is correct.)

Example 2: Find the z-transform for

$$x(k) = (3 + 4\cdot 0.9^k \cos(0.2k+0.3))u(k)$$

Solution:

$$X(z) = 3\left(\frac{z}{z-1}\right) + \left(\left(\frac{2\angle 0.3}{z-0.9\angle 0.2}\right) + \left(\frac{2\angle-0.3}{z-0.9\angle-0.2}\right)\right)$$

Example 3: Find the inverse z-transform for

$$X(z) = \left(\frac{3z}{(z-1)(z-0.9)(z-0.8)}\right)$$

Solution: We're going to need a 'z' in the numerator, so take the partial fraction of

$$X(z) = \left(\frac{3}{(z-1)(z-0.9)(z-0.8)}\right)z = \left(\left(\frac{a}{z-1}\right) + \left(\frac{b}{z-0.9}\right) + \left(\frac{c}{z-0.8}\right)\right)z$$

$$a = \left(\frac{3}{(z-0.9)(z-0.8)}\right)_{z\to 1} = 150$$

$$b = \left(\frac{3}{(z-1)(z-0.8)}\right)_{z\to 0.9} = -300$$

$$c = \left(\frac{3}{(z-1)(z-0.9)}\right)_{z\to 0.8} = 150$$

so

$$X(z) = \left(\left(\frac{150}{z-1}\right) + \left(\frac{-300}{z-0.9}\right) + \left(\frac{150}{z-0.8}\right)\right)z$$

$$X(z) = \left(\frac{150z}{z-1}\right) + \left(\frac{-300z}{z-0.9}\right) + \left(\frac{150z}{z-0.8}\right)$$

From our table

$$x(k) = \left(150 - 300 \cdot (0.9)^k + 150 \cdot (0.8)^k\right) \cdot u(k)$$

Example 4: Find the inverse z-transform for

$$X(z) = \left(\frac{3}{(z-0.9)(z^2-1.8z+0.9)}\right)$$

Solution: We're going to need a 'z' in the numerator, so take the partial fraction of

$$X(z) = \left(\frac{3}{z(z-0.9)(z-0.9487\angle 0.3218)(z-0.9487\angle -0.3218)}\right)z$$

$$= \left(\left(\frac{a}{z}\right) + \left(\frac{b}{z-0.9}\right) + \left(\frac{c}{z-0.9487\angle 0.3218}\right) + \left(\frac{d}{z-0.9487\angle -0.3218}\right)\right)z$$

$$a = \left(\frac{3}{(z-0.9)(z-0.9487\angle 0.3218)(z-0.9487\angle -0.3218)}\right)_{z\to 0} = -3.7037$$

$$b = \left(\frac{3}{z(z-0.9487\angle 0.3218)(z-0.9487\angle -0.3218)}\right)_{z\to 0.9} = 37.037$$

$$c = \left(\frac{3}{z(z-0.9)(z-0.9487\angle -0.3218)} \right)_{z=0.9487\angle 0.3218} = 17.5682\angle 2.8198$$

$$d = \left(\frac{3}{z(z-0.9)(z-0.9487\angle 0.3218)} \right)_{z=0.9487\angle -0.3218} = 17.5682\angle -2.8198$$

so

$$X(z) = \left(\left(\frac{-3.7037}{z} \right) + \left(\frac{37.037}{z-0.9} \right) + \left(\frac{17.5682\angle 2.8198}{z-0.9487\angle 0.3218} \right) + \left(\frac{17.5682\angle -2.8198}{z-0.9487\angle -0.3218} \right) \right) z$$

$$X(z) = -3.7037 + \left(\frac{37.037z}{z-0.9} \right) + \left(\frac{z17.5682\angle 2.8198}{z-0.9487\angle 0.3218} \right) + \left(\frac{z17.5682\angle -2.8198}{z-0.9487\angle -0.3218} \right)$$

$$x(k) = \left(-3.7037\delta(k) + 37.037(0.9)^k + 35.1724 \cdot (0.9487)^k \cos(0.3128k + 2.8198) \right) \quad \mathrm{u(k)}$$