Properties of z-Transform

Properties of z-Transforms

Linearity: $ax(k) + by(k) \leftrightarrow aX(z) + bY(z)$

Time Delay: $x(k-1) \leftrightarrow \left(\frac{1}{z}\right)X(z)$

Convolution: $x(k) ** y(k) = X(z)Y(z)$

Initial Value: $x(0) = \lim_{z\to\infty} (X(z))$

Final Value: $x(\infty) = \lim_{z\to 1} ((z-1)X(z))$

Proofs

Linearity: $ax(k) + by(k) \leftrightarrow aX(z) + bY(z)$

Proof:

$$Z(ax(k) + by(k)) = \sum_{n=0}^{\infty} z^{-n}(ax(n) + by(n))$$

$$= \sum_{n=0}^{\infty} z^{-n}ax(n) + z^{-n}by(n)$$

$$= a\sum_{n=0}^{\infty} z^{-n}x(n) + b\sum_{n=0}^{\infty} y(n)$$

$$= aX(z) + bY(z)$$

Time Delay: $x(k-1) \leftrightarrow z^{-1}X(z)$

Proof:

$$Z(x(k-1)) = \sum_{n=0}^{\infty} z^{-n}(x(n-1))$$

Do a change in variable:

m = n-1

$$= \sum_{m=-1}^{\infty} z^{-(m+1)}x(m) = \sum_{m=0}^{\infty} z^{-1}z^{-m}x(m)$$

$$= z^{-1}X(z)$$

Convolution: $\sum_{n=0}^{\infty} x(k-n)y(n) = X(z)Y(z)$

Proof :

$$X(z)Y(z) = (x_0 + z^{-1}x_1 + z^{-2}x_2 + z^{-3}x_3 + ...)(y_0 + z^{-1}y_1 + z^{-2}y_2 + z^{-3}y_3 + ...)$$

$$= \sum_k x_{n-k} y_k z^{-n}$$

$$= x(k) * *y(k)$$

Initial Value: $x(0) = \lim_{z \to \infty} (X(z))$

Proof: Express x(k) as

$$x(k) = x(0) + \left(\frac{x(1)}{z}\right) + \left(\frac{x(2)}{z^2}\right) + \left(\frac{x(3)}{z^3}\right) + ...$$

As $z \to \infty$

$$x(k) = x(0)$$

Final Value: $x(\infty) = \lim_{z \to 1} ((z-1)X(z))$

Proof: Do a partial fraction expansion of X(z)

$$X(z) = \left(\frac{a}{z-1}\right) + \left(\frac{b}{z-c}\right) + ...$$

The inverse-z transform will be

$$x(k) = a + b \cdot c^k + ...$$

As k goes to infinity, all terms go to zero (assuming the system is stable) except for the first term. The way you find 'a' using partial fractions is

$$a = \lim_{z \to 1} ((z-1) \cdot X(z))$$

z-Transforms and Markov Chains

A Markov chain is a discrete-time function of the form

$$x(k+1) = Ax(k)$$

with an initial condition, $x(0)$. For example, assume three people are tossing a ball around. Every 1 second, the ball is tossed:

- Player #1
  - Keeps the ball 10% of the time
  - Passes it to player #2 50% of the time, and
  - Passes to player #3 40% of the time.
- Player #2
  - Passes the ball to player #1 80% of the time, and
  - Passes the ball to player #3 20% of the time.
- Player #3
  - Passes the ball to player #2 20% of the time, and
  - Keeps the ball 80% of the time.

A description of this game is

$$x(k+1) = \begin{bmatrix} 0.1 & 0.8 & 0 \\ 0.5 & 0 & 0.2 \\ 0.4 & 0.2 & 0.8 \end{bmatrix} x(k)$$

Note that the columns add up to 1.00: every second the ball goes somewhere with a probability of 1.00. The entries for the 1st column are where the ball goes if player #1 has the ball, the second column is where the ball winds up if player #2 has the ball, etc.

In terms of z-transforms:

$$zX = \begin{bmatrix} 0.1 & 0.8 & 0 \\ 0.5 & 0 & 0.2 \\ 0.4 & 0.2 & 0.8 \end{bmatrix} X$$

The final value theorem tells you who has the ball as time goes to infinity:

$$X = \begin{bmatrix} 0.1 & 0.8 & 0 \\ 0.5 & 0 & 0.2 \\ 0.4 & 0.2 & 0.8 \end{bmatrix} X$$

meaning X is the eigenvector associated with the eigenvalue of 1.000

```
>> [m,v] = eig(A)

  0.7636   -0.5491    0.2883
 -0.6322   -0.2487    0.3243
 -0.1314    0.7979    0.9009


 -0.5623         0         0
       0    0.4623         0
       0         0    1.0000
```

The third eigenvector is proportional to the probability that any given player has the ball. All probabilities have to add to one, however. So.

```
>> V = m(:,3)

  0.2883
  0.3243
  0.9009

>> V / sum(V)

  0.1905
  0.2143
  0.5952
```

As k goes to infinity

- The probability that player #1 has the ball is 0.1905
- The probability that player #2 has the ball is 0.2143
- The probability that player #3 has the ball is 0.5952

You can also get this result by passing the ball a large number of times

```
>> A^1000

  0.1905    0.1905    0.1905
  0.2143    0.2143    0.2143
  0.5952    0.5952    0.5952
```

Example 2: Two people are playing tennis. Player #1 has a 60% change of winning any given game. The match is over when one player is up two games. What is the probability that player #1 will win the match?

Solution: This too is a Markov chain. If you define the states to be

$$X = \begin{bmatrix} x_2 \\ x_1 \\ x_0 \\ x_{-1} \\ x_{-2} \end{bmatrix} = \begin{bmatrix} \text{up 2 games} \\ \text{up 1 game} \\ \text{tied} \\ \text{down 1 game} \\ \text{down 2 games} \end{bmatrix}$$

then

$$X_{k+1} = \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} X_k$$

(If you're up one game after k games (column #2), then

- There is a 60% chance you'll be up 2 games after the next game
- There is a 40% chance you'll be even after the next game.

From the final value theorem:

$$(zX)_{z=1} = \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} X_k$$

The eigenvector associated with $\lambda = 1$ is

$$\Lambda = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which doesn't help: the final value is either you win (+2 games) or you lose (-2 games), or some combination thereof. Another solution is to play the game a bunch of times (like 1000 times)

```
>> A^1000

    1.0000    0.8769    0.6923    0.4154         0
         0    0.0000         0    0.0000         0
         0         0    0.0000         0         0
         0    0.0000         0    0.0000         0
         0    0.1231    0.3077    0.5846    1.0000
```

If you start out even (column #3),

- Player #1 has a 69.23% chance of winning the match
- Player #2 has a 30.77% chance of winning the match

If you give odds by giving player #2 a win (column #3)

- Player #1 has a 41.54% chance of winning
- Player #2 has a 58.46% chance of winning.

z-Transform and Moment Generating Functions

In statistics, they're called *Moment Generating Functions*. In ECE, they're called *z-tansforms.*

Problem:

- Let X be the number of times you have to roll a 10-sided die until you roll a 1.
- Then, roll a 4-sided die until you get X ones.
- Let N be the total number of times you roll the dice.

What is the probability distribution of N?

Solution: If you solve in the time-domain (or k domain), you need to use convolution. If you solve in the z-domain, you use multiplication.

The probability distribution of x(k) is

$$x(k) = 0.1 \cdot 0.9^{k-1} \qquad k > 0$$

This is an exponential distribution with a z-transform (or moment generating function if you're a statistics major) of

$$X(z) = \left(\frac{0.1}{z-0.9}\right)$$

The probability distribution of y(k) is

$$y(k) = 0.25 \cdot (0.75)^{k-1} \qquad k > 0$$

which has the z-transform of

$$Y(z) = \left(\frac{0.25}{z-0.75}\right)$$

The convolution of x(k) and y(k) is the product of X(z) and Y(z)

$$N(z) = X(z) \cdot Y(z)$$

$$N = \left(\frac{0.1}{z-0.9}\right)\left(\frac{0.25}{z-0.75}\right)$$

Taking the partial fraction expansion

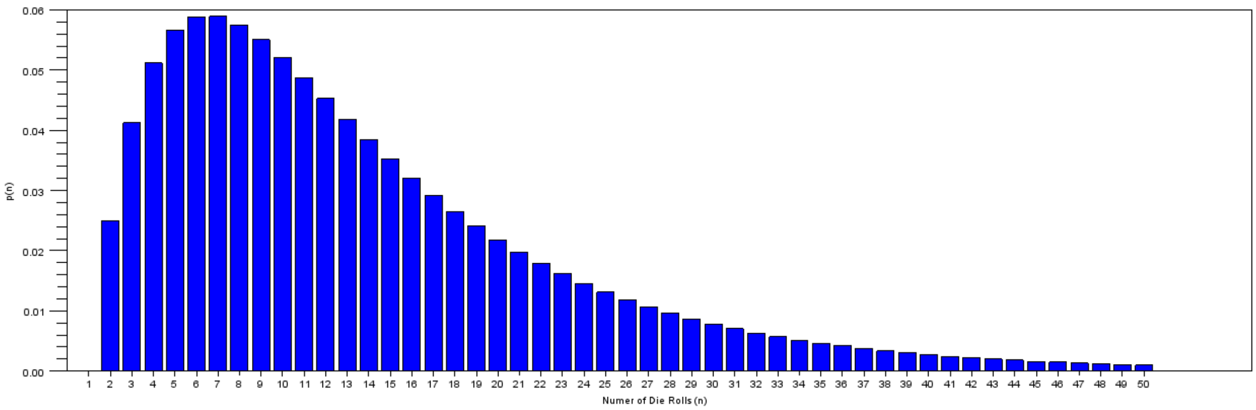
$$N = \left(\frac{0.1666}{z-0.9}\right) + \left(\frac{-0.1666}{z-0.75}\right)$$

resulting in

$$N = \left(\frac{1}{z}\right)\left(\left(\frac{0.1666z}{z-0.9}\right) + \left(\frac{-0.1666z}{z-0.75}\right)\right)$$

$$n(k) = \left(\frac{1}{z}\right)\left(\frac{0.9^k - 0.75^k}{6}\right)u(k)$$

$$n(k) = \left(\frac{0.9^{k-1} - 0.75^{k-1}}{6}\right)u(k-1)$$



n(k): Probability of Rolling the dice k times