ECE 376 - Homework #10

z-Transforms and Digital Filters.   Due Monday, November 15th

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{20(s+2)}{(s+1)(s+3)(s+10)}\right)X$$

a) What is the differential equation relating X and Y?

multiply out and cross multiply

$$Y = \left(\frac{20s+40}{s^3+14s^2+43s+30}\right)X$$

$$(s^3 + 14s^2 + 43s + 30)Y = (20s + 40)X$$

'sY' means 'the derivative of y', so (using prime notation for derivative):

$$y''' + 14y'' + 43y' + 30y = 20x' + 40x$$

$$\frac{d^3y}{dt^3} + 14\frac{d^2y}{dt^2} + 43\frac{dy}{dt} + 30y = 20\frac{dx}{dt} + 40x$$

b) Find y(t) assuming

$$x(t) = 2 + 4\sin(3t)$$

Use superposition and treat this as two separate problems

$x_1(t) = 2$

$$s = 0$$

$$X = 2$$

$$Y = \left(\frac{20s+40}{s^3+14s^2+43s+30}\right)_{s=0}(2) = 2.667$$

$$y_1(t) = 2.667$$

$x_2(t) = 4\sin(3t)$

$$s = j3$$

$$X = 0 - j4$$

real = cosine, imag = -sine

$$Y = \left(\frac{20s+40}{s^3+14s^2+43s+30}\right)_{s=j3} \cdot (0 - j4) = -2.0061 - j0.4648$$

$$y_2(t) = -2.0061\cos(3t) + 0.4648\sin(3t)$$

$$y(t) = y_1 + y_2 = 2.667 - 2.0061\cos(3t) + 0.4648\sin(3t)$$

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.2(z+1)}{(z-0.98)(z-0.96)}\right)X$$

a) What is the difference equation relating X and Y?

multiply out and cross multiply

$$Y = \left(\frac{0.2z+0.2}{z^2-1.94z+0.9408}\right)X$$

$$(z^2 - 1.94z + 0.9408)Y = (0.2z + 0.2)X$$

'zY' means 'y(k+1)' or 'the next value of y(k)'

$$y(k+2) - 1.94y(k+1) + 0.9408y(k) = 0.2x(k+1) + 0.2x(k)$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2 + 3\sin(4t)$$

Use superposition and treat this as two separate problems

$$x_1(t) = 2$$

$$s = 0$$

$$z = e^{sT} = 1$$

$$Y = \left(\frac{0.2(z+1)}{(z-0.98)(z-0.96)}\right)_{z=1} \cdot (2) = 1000$$

$$y_1(t) = 1000$$

$$x_2(t) = 3\sin(4t)$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04} = 1\angle 2.292^0$$

$$X = 0 - j3$$ *real = cosine, imag = -sine*

$$Y = \left(\frac{0.2(z+1)}{(z-0.98)(z-0.96)}\right)_{z=1\angle 2.292^0} \cdot (0 - j3)$$

$$Y = -457.27 + j155.47$$

$$y_2(t) = -457.27\cos(4t) - 155.47\sin(4t)$$

The total answer is the sum of the two

$$y(t) = 1000 - 457.27\cos(4t) - 155.47\sin(4t)$$

Problem 3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{1000}{(s+3)(s+7)(s+20)}\right)$$

3) Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

$$s_1 = -3 \qquad z_1 = e^{sT} = e^{(-3)(0.01)} = 0.9704$$

$$s_2 = -7 \qquad z_2 = e^{-0.07} = 0.9324$$

$$s_3 = -20 \qquad z_3 = e^{-0.2} = 0.8187$$

meaning

$$G(z) = \left(\frac{k}{(z-0.9704)(z-0.9324)(z-0.8187)}\right)$$

The DC gain of G(s) is

$$\left(\frac{1000}{(s+3)(s+7)(s+20)}\right)_{s=0} = 2.3810$$

Pick 'k' so that G(z) has the same DC gain

$$\left(\frac{k}{(z-0.9704)(z-0.9324)(z-0.8187)}\right)_{z=1} = 2.3810$$

$$k = 0.00086235$$

so...

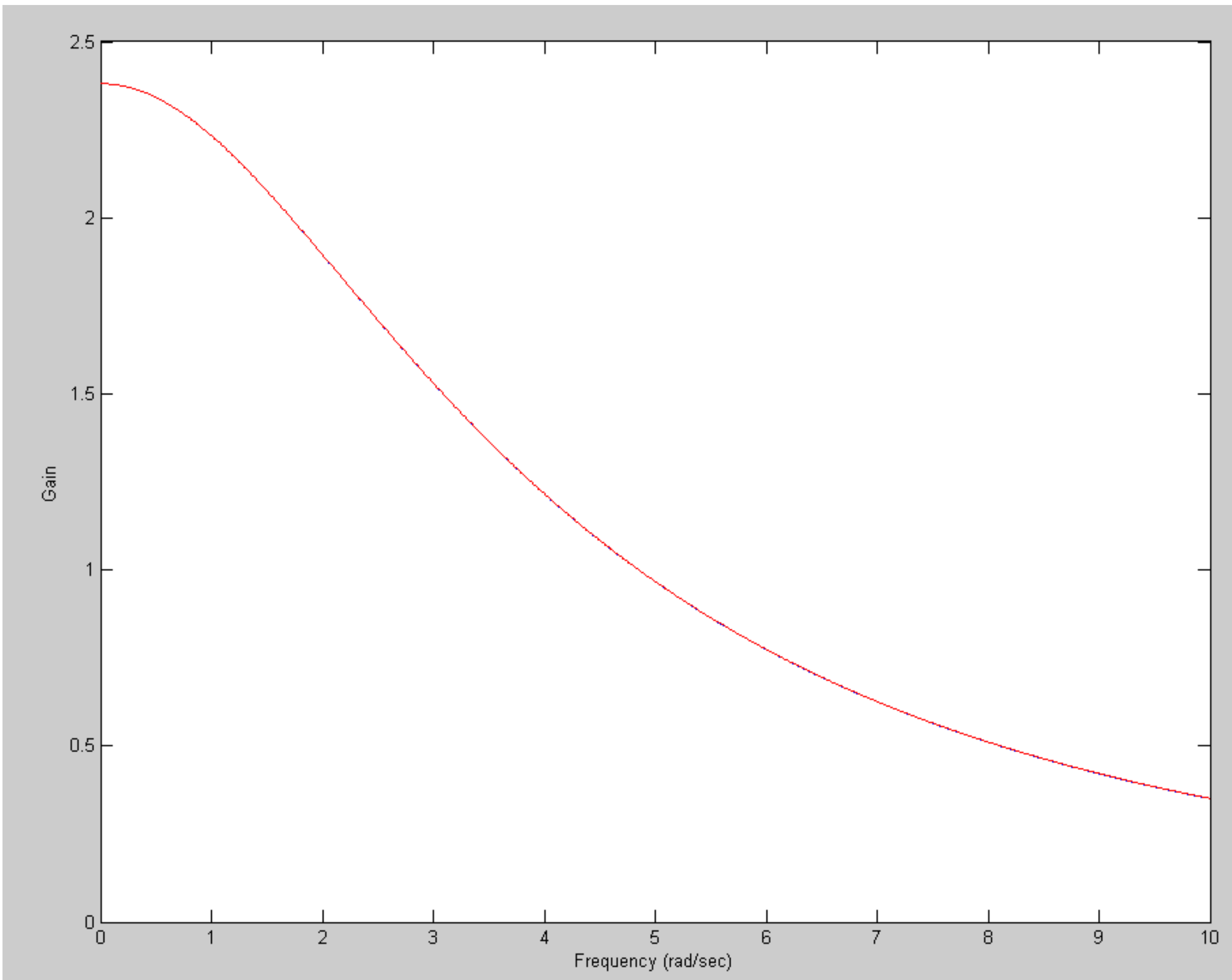
$$G(z) = \left(\frac{0.00086235}{(z-0.9704)(z-0.9324)(z-0.8187)}\right)$$

Plotting the gain vs. frequency

```
>> w = [0:0.01:10]';
>> s = j*w;
>> Gs = 1000 ./ ( (s+3).*(s+7).*(s+20) );

>> T = 0.01;
>> z = exp(s*T);
>> Gz = 0.00086235 ./ ( (z-0.9704).*(z-0.9324).*(z-0.8187) );

>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
```



G(s) (blue) and G(z) (red).

Problem 4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{6s}{(s+3+j20)(s+3-j20)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

$$s_1 = 0 \qquad z_1 = e^{sT} = e^0 = 1$$

$$s_2 = -3 + j20 \qquad z_2 = e^{sT} = 0.9511 + j0.1928$$

$$s_3 = -3 - j20 \qquad z_3 = e^{sT} = 0.9511 - j0.1928$$

So, G(z) looks like

$$G(z) = \left(\frac{k(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)}\right)$$

To find 'k', match the gain somewhere. DC doesn't work since the DC gain is zero. Pick some other frequency, such as 20 rad/sec

$$\left|\left(\frac{6s}{(s+3+j20)(s+3-j20)}\right)_{s=j20}\right| = 0.9972$$

Pick 'k' so that the gain matches

$$z = e^{sT} = e^{j0.2}$$

$$\left|\left(\frac{k(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)}\right)_{z=e^{j0.2}}\right| = 0.9972$$

$$k = 0.05794$$

meaning

$$G(z) = \left(\frac{0.05794(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)}\right)$$

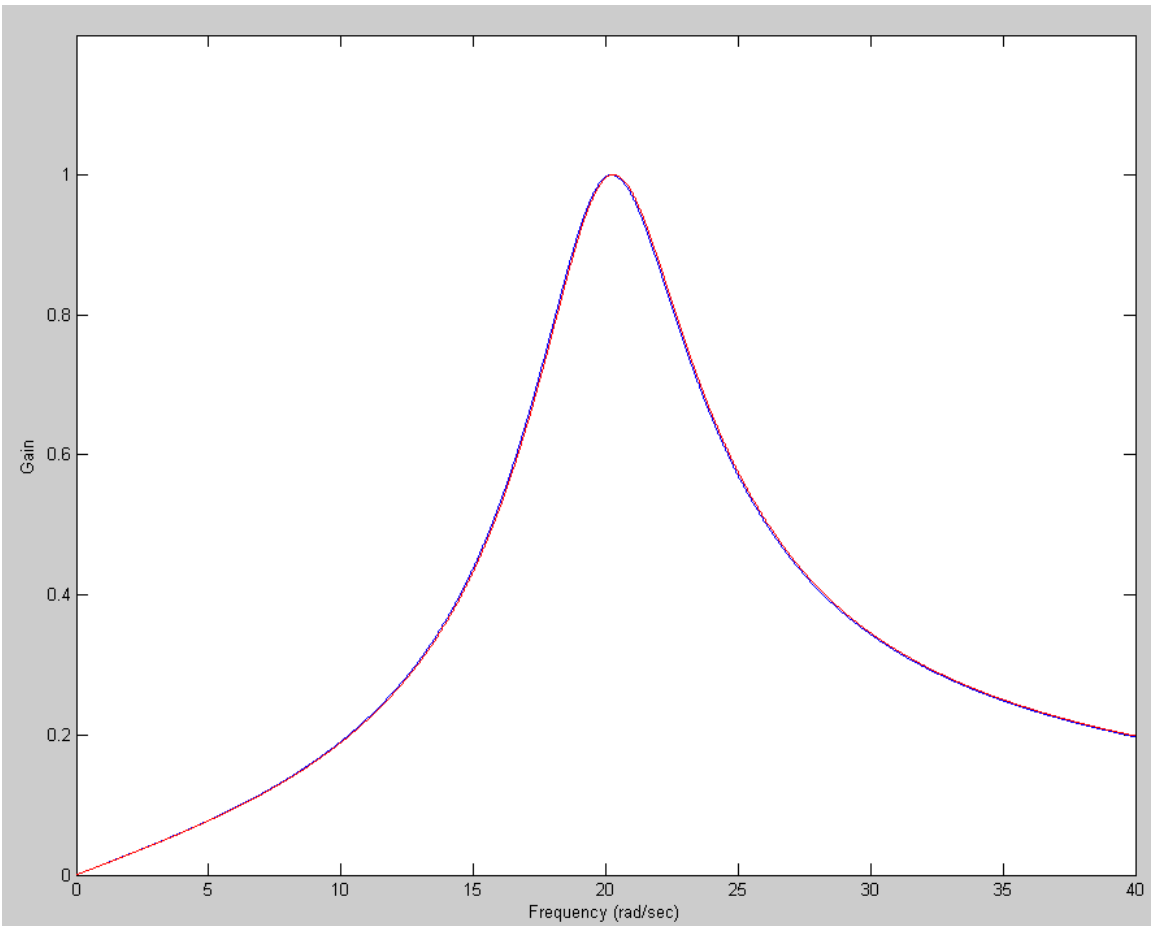
or

$$G(z) = \left(\frac{0.05794(z-1)}{z^2-1.902z+0.9418}\right)$$

Plotting the gain vs. frequency for the two:

```
>> w = [0:0.01:40]';
>> s = j*w;
>> Gs = 6*s ./ ( (s+3+j*20).*(s+3-j*20) );
>>
>> T = 0.01;
>> z = exp(s*T);
>> Gz = 0.05794*(z-1) ./ ( z.^2 - 1.902*z + 0.9418 );

>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
>> xlabel('Frequency (rad/sec)')
>> ylabel('Gain');
```



G(s) (red) & G(z) (blue)

Problem 5) Write a C program to implement the digital filter, G(z)

```
        0.05794 (z-1)
  ----------------------
  (z^2 - 1.902z + 0.9418)
```

$$Y = \left(\frac{0.05794(z-1)}{z^2-1.902z+0.9418}\right)X$$

Rewrite as

$$(z^2 - 1.902z + 0.9418)Y = 0.05794(z-1)X$$

$$z^2 Y = (1.902z - 0.9418)Y + 0.05794(z-1)X$$

$$y(k+2) = 1.902\ y(k+1) - 0.9418\ y(k) + 0.05794\ (\ x(k+1) - x(k)\ )$$

Shift time by 2 (change of variable)

$$y(k) = 1.902\ y(k-1) - 0.9418\ y(k-2) + 0.05794\ (\ x(k-1) - x(k-2)\ )$$

That's essentially the C program:

```
while(1) {

  x2 = x1;
  x1 = x0;
  x0 = A2D_Read(0);

  y2 = y1;
  y1 = y0;
  y0 = 1.902*y1 - 0.9418*y2 + 0.05794*( x1 - x2 );

  D2A(y0);

  Wait_10ms();

  }
```