ECE 376 - Homework #11

z-Transforms and Digital Filters.   Due Monday, November 21st

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{10(s+3)}{(s^2+2s+25)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$(s^2+2s+25)Y = (10s+30)X$$

sY means *the derivative of y*

$$\boxed{y'' + 2y' + 25y = 10x' + 30x}$$

or

$$\boxed{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 25y = 10\frac{dx}{dt} + 30x}$$

b) Find y(t) assuming

$$x(t) = 2 + 4\sin(5t)$$

Note that x(t) has been on for all time. This means use phasor analysis. Treat this as two problems:

- One at DC: x(t) = 2
- One at 5 rad/sec: x(t) = 4 sin(5t)

x(t) = 2

$$s = 0$$

$$X = 2$$

$$Y = \left(\frac{10(s+3)}{(s^2+2s+25)}\right)_{s=0} \cdot (2) = 2.4$$

x(t) = 4 sin(5t)

$$s = j5$$

$$X = 0 - j4$$

$$Y = \left(\frac{10(s+3)}{(s^2+2s+25)}\right)_{s=j5} \cdot (0 - j4) = -12 - j20$$

$$y(t) = -12\cos(5t) + 20\sin(5t)$$

The total answer is the DC term plus the AC term

$$\boxed{y(t) = 2.4 - 12\cos(5t) + 20\sin(5t)}$$

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.01(z+0.8)}{(z-0.95)(z-0.88)}\right)X$$

a) What is the difference equation relating X and Y?

Cross multiply

$$(z^2 - 1.83z + 0.836)Y = 0.01(z+0.8)X$$

zX means *the next value of x, or x(k+1)*

$$\boxed{y(k+2) - 1.83y(k+1) + 0.836y(k) = 0.01(x(k+1) + 0.8x(k))}$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2 + 4\sin(5t)$$

Use superposition

x(t) = 2:

$$s = 0$$

$$z = e^{sT} = 1$$

$$X = 2$$

$$Y = \left(\frac{0.01(z+0.8)}{(z-0.95)(z-0.88)}\right)_{z=1} \cdot (2) = 6$$

$$y(t) = 6$$

x(t) = 4 sin(5t)

$$s = j5$$

$$z = e^{sT} = e^{j0.05} = 1\angle 2.865^0$$

$$X = 0 - j4$$

$$Y = \left(\frac{0.01(z+0.8)}{(z-0.95)(z-0.88)}\right)_{z=1\angle 2.865^0} \cdot (0 - j4)$$

$$Y = -7.3627 - j3.1334$$

$$y(t) = -7.3627\cos(5t) + 3.1334\sin(5t)$$

The total answer is DC + AC

$$\boxed{y(t) = 6 - 7.3627\cos(5t) + 3.1334\sin(5t)}$$

Problem 3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{90}{(s+3)(s+7)(s+11)}\right)$$

3) Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

Convert from the s-plane to the z-plane using

$$s = -3 \qquad z = e^{sT} = e^{-0.03} = 0.9704$$

$$s = -7 \qquad z = e^{sT} = e^{-0.07} = 0.9324$$

$$s = -11 \qquad z = e^{sT} = e^{=-/0.11} = 0.8958$$

so

$$G(z) \approx \left(\frac{k}{(z-0.9704)(z-0.9324)(z-0.8958)}\right)$$

Pick k to match the DC gain

$$G_s(s=0) = \left(\frac{90}{(s+3)(s+7)(s+11)}\right)_{s=0} = 0.3896$$

$$G_z(z=1) = \left(\frac{k}{(z-0.9704)(z-0.9324)(z-0.8958)}\right)_{z=1} = 0.3869$$

$$k = 8.109 \cdot 10^{-5}$$

Add two zeros at z = 0 to match the phase at 0.1 rad/sec

$$\boxed{G(z) = \left(\frac{8.109 \cdot 10^{-5}}{(z-0.9704)(z-0.9324)(z-0.8958)}\right)}$$

In Matlab:

```
>> Ps = [-3, -7, -11]
Ps =     -3     -7    -11

>> T = 0.01;
>> Pz = exp(s*T)
Pz =     0.9704     0.9324     0.8958

>> s = 0;
>> DC = 90 / ( (s+3)*(s+7)*(s+11) )
DC =     0.3896

>> z = 1;
>> k = DC * ( (z+0.9704)*(z+0.9324)*(z+0.8958) )
k =  8.1090e-005

>> s = j*0.1;
>> Gs = 90 / ( (s+3)*(s+7)*(s+11) )
Gs =   0.3887 - 0.0221i

>> z = exp(s*T);
>> Gz = k / ( (z+0.9704)*(z+0.9324)*(z+0.8958) )
Gz =   0.3887 - 0.0227i
```

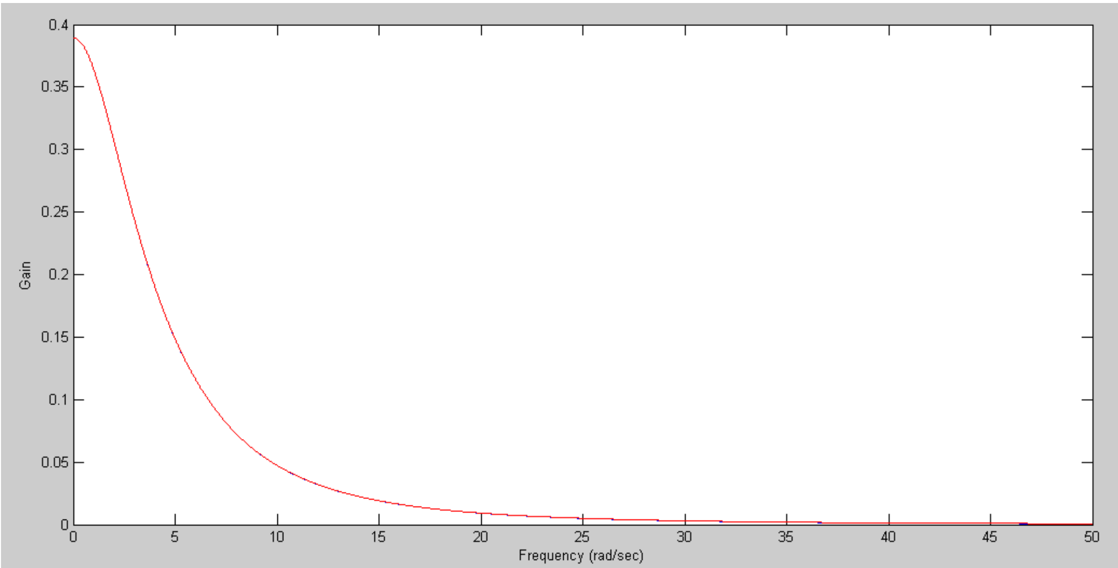
To plot the gain vs. frequency

```
>> w = [0:0.01:50]';

>> s = j*w;
>> Gs = 90 ./ ( (s+3).*(s+7).*(s+11) );

>> z = exp(s*T);
>> Gz = 8.109e-5 ./ ( (z-0.9704) .* (z-0.9324) .* (z-0.8958) );

>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain');
```



Problem 4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{6s}{(s+1+j20)(s+1-j20)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

```
>> Zs = 0;
>> Ps = [-1+j*20,-1-j*20];
>> T = 0.01;
>> Zz = exp(Zs*T)

Zz =     1

>> Pz = exp(Ps*T)

Pz =   0.9703 + 0.1967i   0.9703 - 0.1967i

>> s = j*20;
>> Gs = 6*s / ( (s+1+j*20) * (s+1-j*20) )

Gs =   2.9981 + 0.0750i

>> z = exp(s*T);
>> Gz = (z-1) / ( (z-Pz(1)) * (z-Pz(2)) )

Gz =  50.5959 - 3.8191i

>> k = abs(Gs) / abs(Gz)

k =    0.0591

>> poly(Pz)

ans =    1.0000   -1.9406    0.9802
```

meaning

$$G(z) = 0.0591\left(\frac{z-1}{z^2-1.9406z+0.9802}\right)$$

Plotting

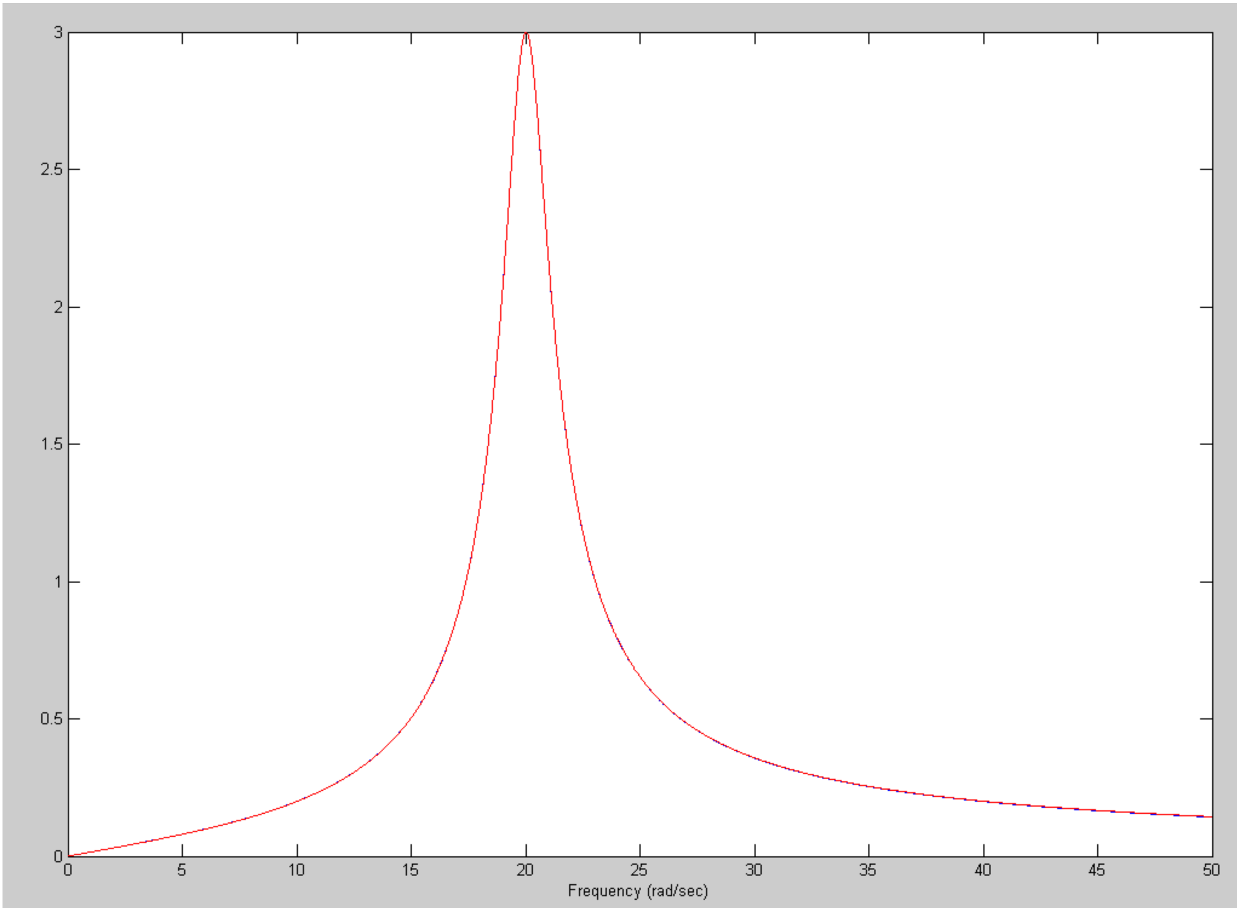
```
>> w = [0:0.01:50]';

>> s = j*w;
>> Gs = 6*s ./ ( (s+1+j*20).*(s+1-j*20) );

>> T = 0.01;
>> z = exp(s*T);
>> Gz = k*(z-1) ./ ( (z-Pz(1)).*(z-Pz(2)) );

>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('Frequency (rad/sec)');
```

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.



Gain of Gs(s) (red), Gain of Gz(z) (blue)

Problem 5) Write a C program to implement the digital filter, G(z)

$$Y = 0.0591\left(\frac{z-1}{z^2-1.9406z+0.9802}\right)X$$

$$(z^2 - 1.9406z + 0.9802)Y = 0.0591(z-1)X$$

$$z^2 Y = 1.9406zY + 0.9802Y + 0.0591(z-1)X$$

y(k+2) = 1.9406 y(k+1) + 0.9802 y(k) + 0.0591*(x(k+1) - x(k))

Shift by two (change of variable)

y(k) = 1.9406 y(k-1) + 0.9802 y(k-2) + 0.0591*(x(k-1) - x(k-2))

This is your C program

```
while(1) {
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y2 = y1;
   y1 = y0;
   y0 = 1.9406*y1 + 0.9802*y2 + 0.0591*(x1 - x2);

   D2A(y0);

   Wait_10ms();

   }
```