ECE 376 - Homework #11

z-Transforms and Digital Filters.   Due Wednesday, April 20th

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{10(s+3)}{(s+2)(s+7)}\right) X$$

a) What is the differential equation relating X and Y?

cross multiply

$$(s^2 + 9s + 14)Y = (10s + 30)X$$

'sy' means 'the derivative of Y

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 14y = 10\frac{dx}{dt} + 30x$$

$$y'' + 9y' + 14y = 10x' + 30x$$

b) Find y(t) assuming

$$x(t) = 2 + 4\sin(5t)$$

Use superposition treating this as two separate problems (one at each frequency)

(i) $x(t) = 2$

$$s = 0$$

$$Y = \left(\frac{10(s+3)}{(s+2)(s+7)}\right)_{s=0} \cdot (2)$$

$$Y = 4.286$$

(ii) $x(t) = 4\sin(5t)$

$$X = 0 - j4$$ *real = cosine, imag = - sine*

$$s = j5$$

$$Y = \left(\frac{10(s+3)}{(s+2)(s+7)}\right)_{s=j5} \cdot (0 - j4)$$

$$Y = -3.541 - j3.579$$

$$y(t) = -3.541\cos(5t) + 3.579\sin(5t)$$

The total answer is then DC + AC

$$\boxed{y(t) = 4.286 - 3.541\cos(5t) + 3.579\sin(5t)}$$

2) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{0.01(z+1)}{(z-0.98)(z-0.92)}\right)X$$

a) What is the difference equation relating X and Y?

cross multiply and multipy out

$$(z^2 - 1.9z + 0.9016)Y = 0.01(z+1)X$$

'zY' means 'y(k+1)' or 'the next value of y'

$$\boxed{y(k+2) - 1.9y(k+1) + 0.9016y(k) = 0.01(x(k+1) + x(k))}$$

with a change in variable, you could also write this as (either is correct)

$$\boxed{y(k) - 1.9y(k-1) + 0.9016y(k-2) = 0.01(x(k-1) + x(k-2))}$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2 + 4\sin(5t)$$

Use superposition and treat this as two separate problems

i) $x(t) = 2$

$$s = 0$$

$$z = e^{sT} = 1$$

$$Y = \left(\frac{0.01(z+1)}{(z-0.98)(z-0.92)}\right)_{z=1} \cdot (2) = 25$$

ii) $x(t) = 4\sin(5t)$

$X = 0 - j4$ *real = cosine, imag = -sine*

$$s = j5$$

$$z = e^{sT} = 1\angle 0.05rad = 1\angle 2.86^0$$

$$Y = \left(\frac{0.01(z+1)}{(z-0.98)(z-0.92)}\right)_{z=1\angle 2.86^0} \cdot (0 - j4) = -15.799 + j2.901$$

$$y(t) = -15.799\cos(5t) - 2.901\sin(5t)$$

The total answer is DC + AC

$$\boxed{y(t) = 25 - 15.799\cos(5t) - 2.901\sin(5t)}$$

Problem 3) Assume G(s) is a low-pass filter with real poles:

$$G(s) = \left(\frac{90}{(s+2)(s+4)(s+10)}\right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

$$s = -2 \qquad z = e^{sT} = 0.9802$$

$$s = -4 \qquad z = e^{sT} = 0.9608$$

$$s = -10 \qquad z = e^{sT} = 0.9048$$

The form of G(z) is then

$$G(z) = \left(\frac{k}{(z-0.9802)(z-0.9608)(z-0.9048)}\right)$$

To find k, match the DC gain

$$\left(\frac{90}{(s+2)(s+4)(s+10)}\right)_{s=0} = 1.25$$

$$\left(\frac{k}{(z-0.9802)(z-0.9608)(z-0.9048)}\right)_{z=1} = 1.25$$

$$k = 0.000092358$$

so

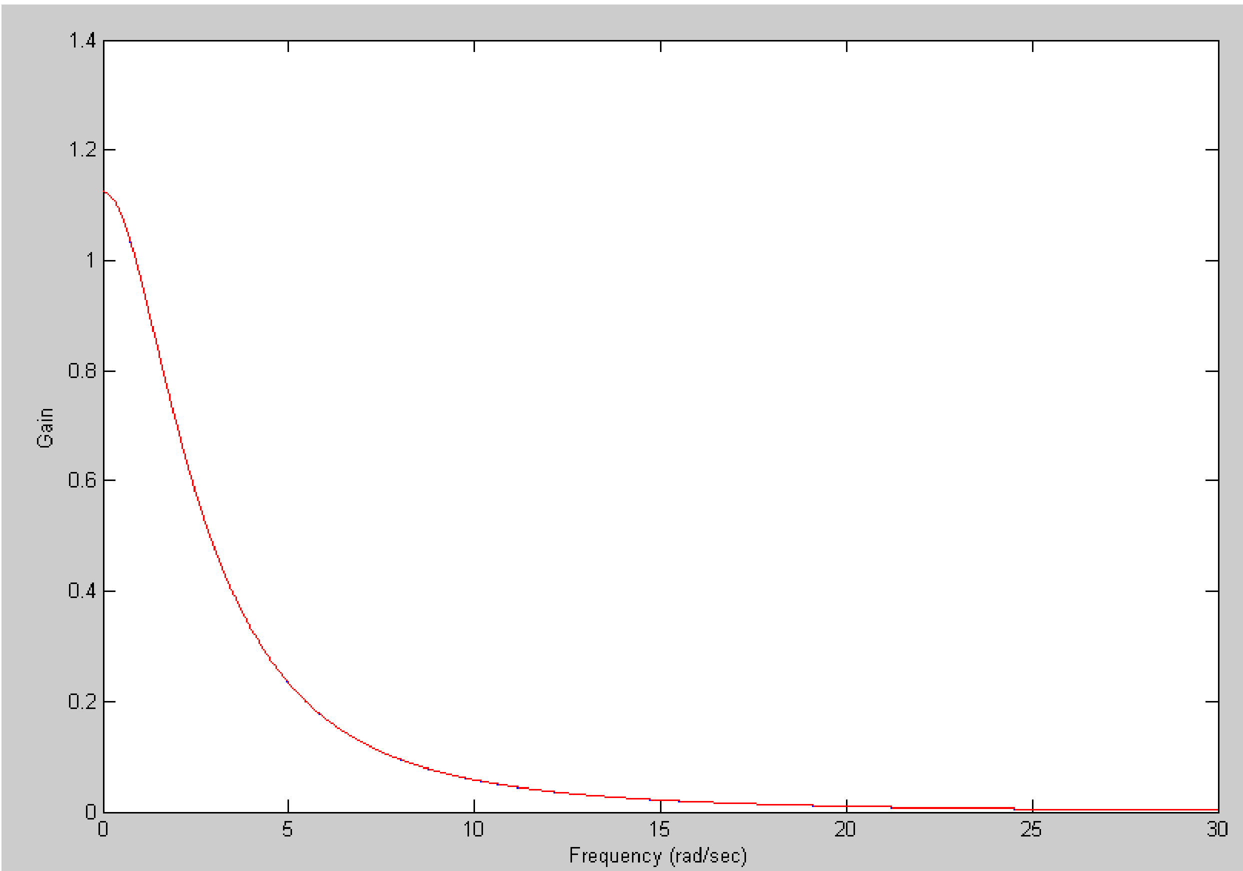
$$G(z) = \left(\frac{9.2358 \cdot 10^{-5}}{(z-0.9802)(z-0.9608)(z-0.9048)}\right)$$

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

In Matlab

```
>> T = 0.01;
>> w = [0:0.01:30]';
>> s = j*w;
>> z = exp(s*T);
>>
>> Gs = 90 ./ ( (s+2).*(s+4).*(s+10) );
>>
>> % z-plane
>> p1 = exp(-2*T)

p1 =    0.9802

>> p2 = exp(-4*T)

p2 =    0.9608

>> p3 = exp(-10*T)

p3 =    0.9048

>> Gz = 1 ./ ( (z-p1).*(z-p2).*(z-p3) );
```

```
>> max(abs(Gs))

ans =    1.1250

>> max(abs(Gz))

ans =  1.3534e+004

>> k = max(abs(Gs)) / max(abs(Gz))

k =  8.3122e-005

>> Gz = k ./ ( (z-p1).*(z-p2).*(z-p3) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
>> xlabel('Frequency (rad/sec)');
>> ylabel('Gain');
```



Problem 4) Assume G(s) is the following band-pass filter:

$$G(s) = \left(\frac{6s}{(s+1+j16)(s+1-j16)} \right)$$

Design a digital filter, G(z), which has approximately the same gain vs. frequency as G(s). Assume a sampling rate of T = 0.01 second.

zero

$$s = 0 \qquad z = e^{sT} = 1$$

poles

$$s = -1 + j16 \qquad z = e^{sT} = 0.9774 + j0.1577$$

$$s = -1 - j16 \qquad z = e^{sT} = 0.9744 - j0.1577$$

meaning

$$G(z) = \left(\frac{k(z-1)}{(z-0.9744+j0.1577)(z-0.9744-j0.1577)} \right)$$

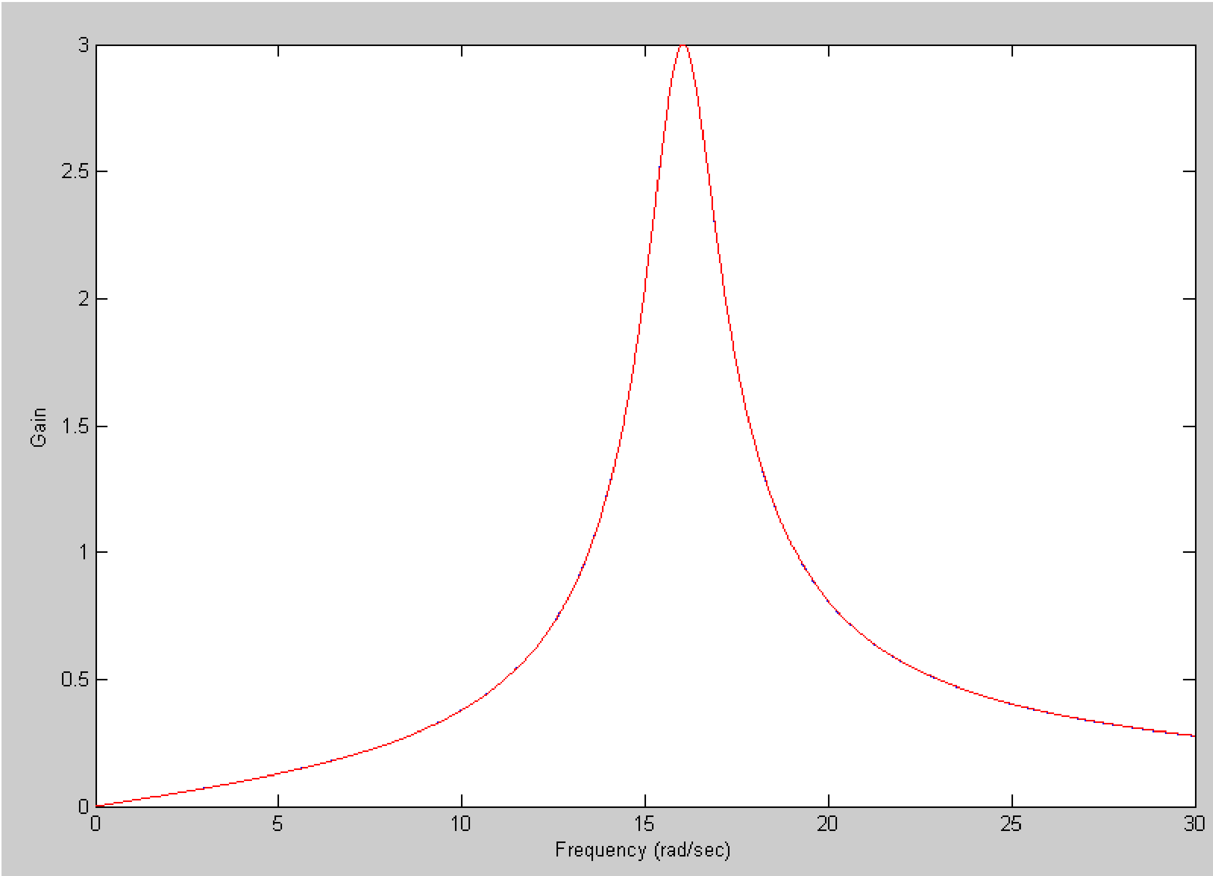
To find k, match the maximum gain vs. frequency (from Matlab)

$$G(z) = \left(\frac{0.0099(z-1)}{(z-0.9744+j0.1577)(z-0.9744-j0.1577)} \right)$$

In Matlab

```
>> T = 0.01;
>> z1 = exp(0*T)

z1 =     1

>> p1 = exp((-1+j*16)*T)

p1 =   0.9774 + 0.1577i

>> p2 = exp((-1-j*16)*T)

p2 =   0.9774 - 0.1577i

>> w = [0:0.01:30]';
>> s = j*w;
>> z = exp(s*T);
>>
>> Gs = 6*s ./ ( (s+1+j*16).*(s+1-j*16) );
>> Gz = 6*(z-z1) ./ ( (z-p1).*(z-p2) );
>> k = max(abs(Gs)) / max(abs(Gz))

k =    0.0099

>> Gz = k * 6*(z-z1) ./ ( (z-p1).*(z-p2) );
>> plot(w,abs(Gs),'b',w,abs(Gz),'r')
```

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.



Problem 5) Write a C program to implement the digital filter, G(z)

$$G(z) = \left(\frac{0.0099(z-1)}{(z-0.9744+j0.1577)(z-0.9744-j0.1577)} \right)$$

multiply out

```
>> poly([p1,p2])

ans =    1.0000   -1.9548    0.9802
```

$$G(z) = \left(\frac{0.0099(z-1)}{\left(z^2-1.9548x+0.9802\right)} \right)$$

In C:

```
while(1) {

   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y2 = y1;
   y1 = y0;
   y0 = 1.9548*y1 - 0.9802*y2 + 0.0099*(x1 - x2);

   D2A(y0);

   Wait_ms(10);

   }
```