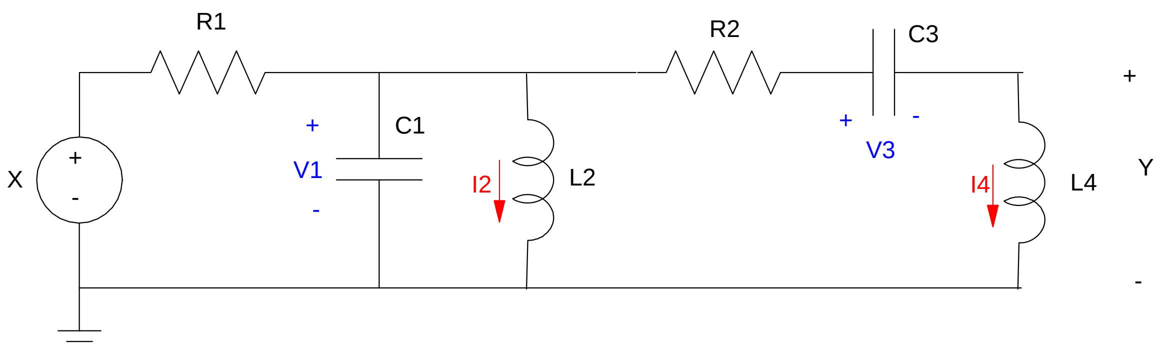
ECE 461 - Solution to Homework Set #5

State-Space, Heat Equation, Wave Equation. - Due Monday, October 5th

1) For the following RLC circuit,



Problem 1: R = 100 Ohms, L = 0.1H, C = 100uF

1a) Express the dynamics of the following circuit in terms of N coupled differential equations.

$$I_1 = C_1 s V_1 = \left(\frac{V_{in} - V_1}{R_1}\right) - I_2 - I_4$$

$$V_2 = L_2 s I_2 = V_1$$

$$I_3 = C_3 s V_3 = I_4$$

$$V_4 = L_4 s I_4 = V_1 - V_3$$

$$y = V_1 - V_3 - I_4 R_2$$

1b) Express the dynamics in state-space form.

$$s\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC_1} & \frac{-1}{C_1} & 0 & \frac{-1}{C_1} \\ \frac{1}{L_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_3} \\ \frac{1}{L_4} & 0 & \frac{-1}{L_4} & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ I_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = \begin{bmatrix} 1 & 0 & -1 & -100 \end{bmatrix}\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ I_4 \end{bmatrix} + [0]V_{in}$$

1c) Find a 2nd-order model with approximately the same behaviour.

In MATLAB, find the transfer function:

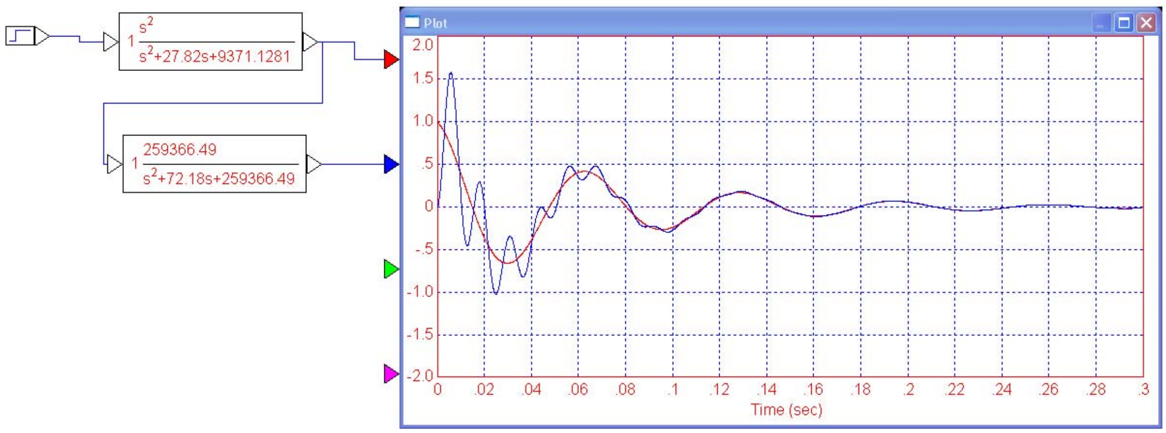
```
-->R = 100;
-->L = 0.1;
-->C = 100e-6;

-->A = [-1/(R*C),-1/C,0,-1/C;1/L,0,0,0;0,0,0,1/C;1/L,0,-1/L,0]

  - 100.   - 10000.    0.    - 10000.
    10.      0.        0.      0.
    0.       0.        0.      10000.
    10.      0.      - 10.     0.

-->B = [1/(R*C);0;0;0]

    100.
    0.
    0.
    0.

-->C = [1,0,-1,100];
-->D = 0;
-->G = ss(A,B,C,D)

--> zpk(G)

               111111 s^2
-----------------------------------------------------------
(s+36.09+j508.08)(s+36.09-j508.08)(s+13.91+j95.83)(s+13.91+j95.83)
```

Keep the dominant pole and drop the fast poles

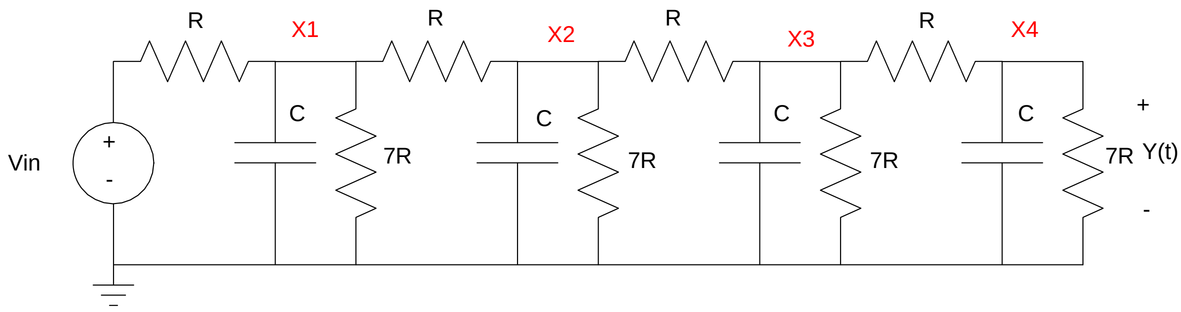
$$G(s) = \left(\frac{259447}{s+36.09 \pm j508.8}\right)\left(\frac{42.8s^2}{s+13.91 \pm j95.83}\right)$$

$$G(s) \approx \left(\frac{42.8s^2}{s+13.91 \pm j95.83}\right)$$

1d) Compare the step responses of your Nth-order system and your 2nd-order model. (scaled by 42.8 so that it plots nicer)



2) For the following RL circuit,



Problem 2: R = 20k Ohms, C = 10uF

2a) Express the dynamics of the following system in terms of N coupled differential equations (i.e. write the voltage node equations).

$$\left(\frac{1}{R}+\frac{1}{R}+\frac{1}{7R}+Cs\right)X_1-\left(\frac{1}{R}\right)V_{in}-\left(\frac{1}{R}\right)X_2=0$$

$$\left(\frac{1}{R}+\frac{1}{R}+\frac{1}{7R}+Cs\right)X_2-\left(\frac{1}{R}\right)X_1-\left(\frac{1}{R}\right)X_3=0$$

$$\left(\frac{1}{R}+\frac{1}{R}+\frac{1}{7R}+Cs\right)X_3-\left(\frac{1}{R}\right)X_2-\left(\frac{1}{R}\right)X_4=0$$

$$\left(\frac{1}{R}+\frac{1}{7R}+Cs\right)X_4-\left(\frac{1}{R}\right)X_3=0$$

2b) Express the dynamics in state-space form.

$$s\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix}=\begin{bmatrix}\frac{-2.143}{RC}&\frac{1}{RC}&0&0\\\frac{1}{RC}&\frac{-2.143}{RC}&\frac{1}{RC}&0\\0&\frac{1}{RC}&\frac{-2.143}{RC}&\frac{1}{RC}\\0&0&\frac{1}{RC}&\frac{-1.143}{RC}\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix}+\begin{bmatrix}\frac{1}{RC}\\0\\0\end{bmatrix}V_{in}$$

$$Y=X_4=\begin{bmatrix}0&0&0&1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\\X_4\end{bmatrix}$$

2c) Find a 2nd-order model with approximately the same behaviour.

in MATLAB:

```
-->A = [-2.143*a,a,0,0;a,-2.143*a,a,0;0,a,-2.143*a,a;0,0,a,-1.143*a]

  - 10.715    5.         0.         0.
    5.      - 10.715     5.         0.
    0.        5.       - 10.715     5.
    0.        0.         5.       - 5.715

-->B = [a;0;0;0]

    5.
    0.
    0.
    0.

-->C = [0,0,0,1]

    0.    0.    0.    1.

-->G = ss(A,B,C,D);

-->zpk(G)
```

$$Y = \left(\frac{625}{(s+1.31)(s+5.71)(s+12.45)(s+18.37)} \right) V_{in}$$

A 2nd-order model would keep the two dominant poles and match the DC gain

```
-->DC = evalfr(G,0)

    0.3626312
```

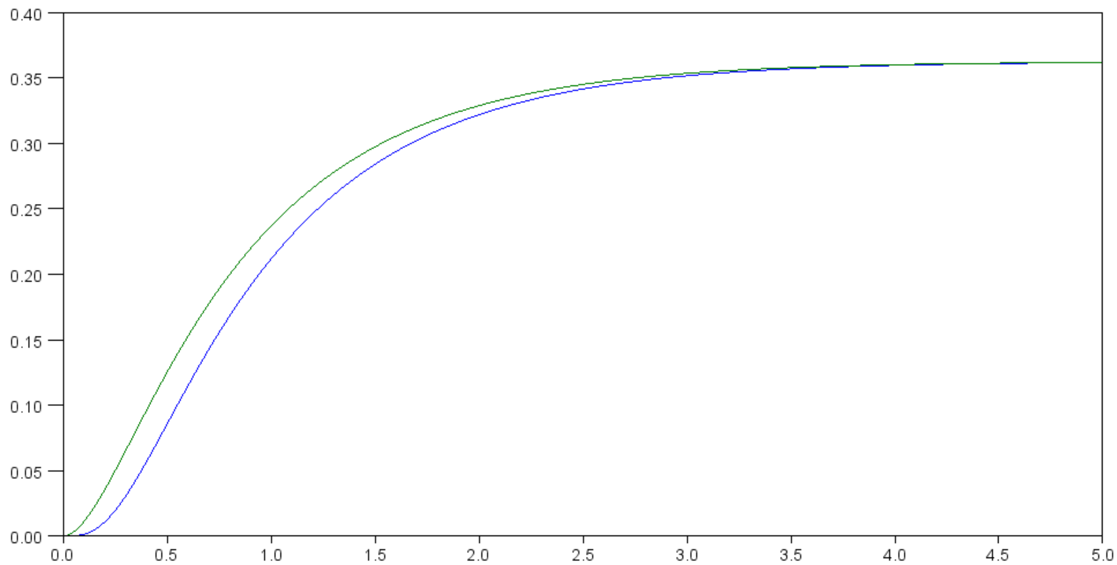
Keep the dominant two poles. Make the numerator and check the DC gain. Adjust so that the DC gain matches

```
-->G2 = zp2ss([],[-1.318,-5.715],1);

-->DC2 = evalfr(G2,0)

    0.1327603

-->k = DC/DC2

    2.7314724

-->G2 = zp2ss([],[-1.318,-5.715],k);
```

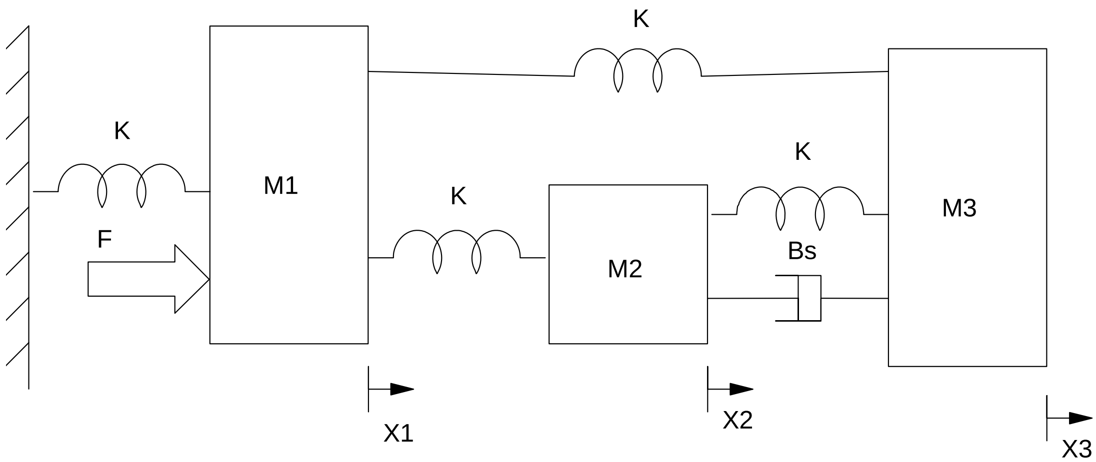
$$Y \approx \left(\frac{2.8414}{(s+1.318)(s+5.715)} \right) V_{in}$$

2d) Compare the step responses of your Nth-order system and your 2nd order model.

```
-->t = [0:0.001:5]';
-->y4 = step(G,t);
-->y2 = step(G2,t);
-->plot(t,y4,t,y2);
```

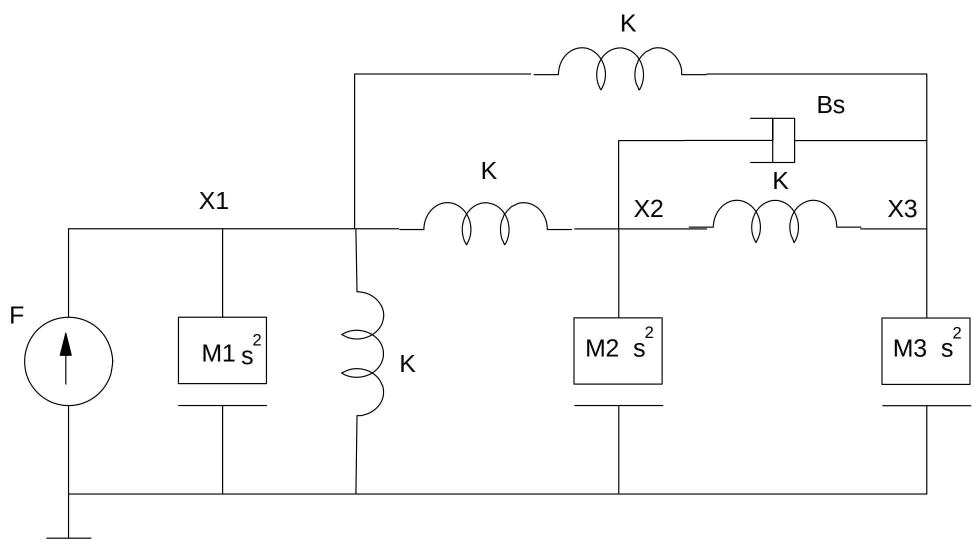


3) For the following mass-spring system,



Problem 3: M = 2kg, K = 10 N/m, B = 0.5 Ns/m

3a) Draw the circuit equivalent for the following mass-spring system



3b) Express the dynamics of the following system in terms of N coupled differential equations

$$(M_1 s^2 + 3K)X_1 - (K)X_2 - (K)X_3 = F$$

$$(M_2 s^2 + Bs + 2K)X_2 - (K)X_1 - (Bs + K)X_3 = 0$$

$$(M_3 s^2 + Bs + 2K)X_3 - (K)X_1 - (Bs + K)X_2 = 0$$

3c) Express the dynamics in state-space form.

$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-3K}{M_1} & \frac{K}{M_1} & \frac{K}{M_1} & 0 & 0 & 0 \\ \frac{K}{M_2} & \frac{-2K}{M_2} & \frac{K}{M_2} & 0 & \frac{-B}{M_2} & \frac{B}{M_2} \\ \frac{K}{M_3} & \frac{K}{M_3} & \frac{-2K}{M_3} & 0 & \frac{B}{M_3} & \frac{-B}{M_3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} F$$

3d) Find a 2nd-order model with approximately the same behaviour.

```
-->a11 = zeros(3,3);
-->a12 = eye(3,3);
-->a21 = [-3*K/M,K/M,K/M;K/M,-2*K/M,K/M;K/M,K/M,-2*K/M]
-->a22 = [0,0,0;0,-B/M,B/M;0,B/M,-B/M]
-->A = [a11,a12;a21,a22]

    0.      0.      0.      1.     0.       0.
    0.      0.      0.      0.     1.       0.
    0.      0.      0.      0.     0.       1.
  - 15.     5.      5.      0.     0.       0.
    5.    - 10.     5.      0.   - 0.25     0.25
    5.      5.    - 10.     0.     0.25   - 0.25

-->B = [0;0;0;1/M;0;0]

    0.
    0.
    0.
    0.5
    0.
    0.

-->C = [0,0,1,0,0,0]
-->D = 0;
-->G = ss(A,B,C,D);
-->zpk(G)
```

$$G(s) = \left(\frac{2.5(s+0.25 \pm j3.86)}{(s \pm j4.319)(s \pm j1.157)(s+0.25 \pm j3.86)} \right)$$

You have two poles on the jw axis, so both are dominant. Taking the one closest to zero

$$G(s) = \left(\frac{4.319^2}{s \pm j4.319} \right) \left(\frac{s+0.25 \pm j3.86}{s+0.25 \pm j3.86} \right) \left(\frac{0.134}{s \pm j1.157} \right)$$

$$G(s) \approx \left(\frac{0.134}{s \pm j1.157} \right)$$

3e) Compare the step responses of your 6th-order system and your 2nd order model.

```
-->G2 = zpk([],p(3:4),00.134)
-->t = [0:0.001:20]';
-->y6 = step(G,t);
-->y2 = step(G2,t);
-->plot(t,y6,t,y2)
```