ECE 461 - Homework #10

z-Transform, Converting G(s) to G(z). Due Monday, November 16th

1a) What is the difference equation the following transfer function represents?

$$Y = \left(\frac{0.01\left(z^2+1.3z+0.4\right)}{z^3-3.2z^2+1.7z-0.62} \right) X$$

Cross multiply

$$(z^3 - 3.2z^2 + 1.7z - 0.62)Y = (0.01(z^2 + 1.3z + 0.4))X$$

Note that zY means y(k+1)

y(k+3) - 3.2y(k+2) + 1.7y(k+1) - 0.62y(k) = 0.01(x(k+2) + 1.3x(k+1) + 0.4x(k))

or if you don't like future, do a change of variable

k+3 = k'

y(k') - 3.2y(k'-1) + 1.7y(k'-2) - 0.62y(k'-3) = 0.01(x(k'-1) + 1.3x(k'-2) + 0.4x(k'-3))

1b) Write a program to implement this filter.

```
while(1) {
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read();

   y3 = y2;
   y2 = y1;
   y1 = y0
   y0 = 3.2*y1 - 1.7*y2 +0.62*y3 + 0.01*(x1 + 1.3*x2 + 0.4*x3 );

   Wait();
   }
```

2a) What is the difference equation the following transfer function represents?

$$Y = \left(\frac{0.03(z+1)(z+0.8)(z+0.6)}{z(z-0.9)(z-0.4)(z-0.2)} \right) X$$

Multiply it out

$$Y = \left(\frac{0.03(z^3+2.4z^2+1.88z+0.48)}{z^4-1.5z^3+0.62z^2-0.072z} \right) X$$

Cross multiply

$$(z^4 - 1.5z^3 + 0.62z^2 - 0.072z)Y = (0.03(z^3 + 2.4z^2 + 1.88z + 0.48))X$$

Note that zY means y(k+1)

y(k+4) - 1.5y(k+3) + 0.62y(k_2) - 0.072y(k+1) = 0.03(x(k+3) + 2.4x(k+2) + 1.88x(k+1) + 0.48x(k)

If you don't like future values, you can do a change of variable

k+4 = k'

y(k') - 1.5y(k'-1) + 0.62y(k'-2) - 0.072y(k'-3) = 0.03(x(k'-1) + 2.4x(k'-2) + 1.88x(k'-3) + 0.48x(k'-4)

Either answer is valid

2b) Write a program to implement this filter.

```
while(1) {
   x4 = x3;
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read();

   y3 = y2;
   y2 = y1;
   y1 = y0
   y0 = 1.5*y1 - 0.62*y2 +.072*y3 +
        0.03*(x1 + 2.4*x2 + 1.88*x3 + 0.48*x4 );

   Wait();
   }
```

Note

- The new filter just changes one line of code
- It doesn't really matter if the poles or zeros are complex. All you care about are the coefficients of the polynomial expansion.

Problem 3-6) Assume a sampling rate of 100ms.

- Determine a filter G(z) which has approximately the same step response as G(s)
- Plot the step response of G(z) and G(s) in VisSim (or similar program) to check your answer.

3) $$G(s) = \left(\frac{30}{(s+2)(s+5)}\right)$$

Convert the poles to the z-plane as $z = e^{sT}$

```
-->s3 = [-2, -5]'

  - 2.
  - 5.

-->z3 = exp(s3*T)

    0.8187308
    0.6065307
```

so

$$G(z) \approx \left(\frac{k}{(z-0.8187)(z-0.6065)}\right)$$

To find the gain on top, match the gain somewhere, like s = 0 (z = +1)

$$\left(\frac{30}{(s+2)(s+5)}\right)_{s=0} = 3$$

$$\left(\frac{k}{(z-0.8187)(z-0.6065)}\right)_{z=1} = 3$$

k = 0.2139717

If you want to match the delay, pick a frequency, such as s = j2 (the corner frequency). Add zeros at z=0 to match the phase shift

$$\left(\frac{30}{(s+2)(s+5)}\right)_{s=j2} = 1.9696\angle-66.80^0$$

$$\left(\frac{0.2139}{(z-0.8187)(z-0.6065)}\right)_{s=j2} = 1.9572\angle-78.92^0$$

Adding zeros at z=0 helps with the phase. Each zero adds

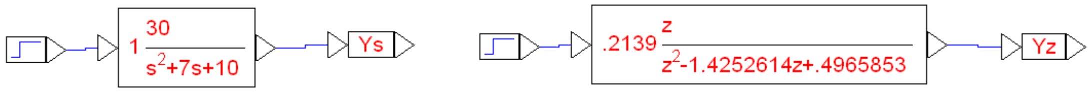
$$(e^{sT})_{s=j2} = 1\angle+11.45^0$$

To fix the phase, you need to add 1.06 zeros

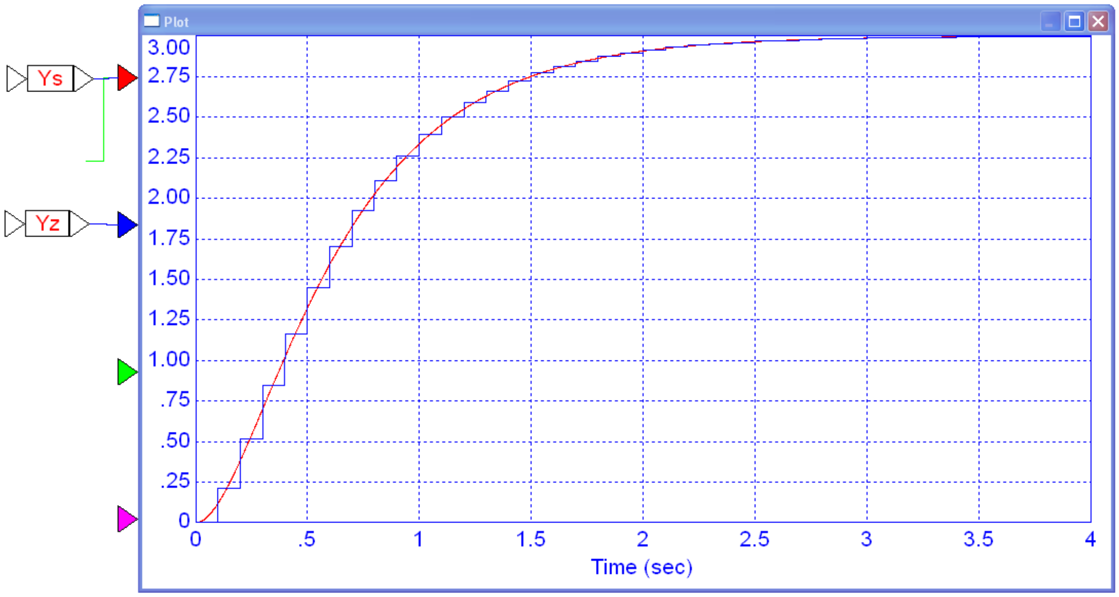
$$n = \frac{78.92^0-66.8^0}{11.45^0} = 1.06$$

Rounding to 1.00 (add one zero at z=0) gives

$$G(z) \approx \left(\frac{0.2139z}{(z-0.8187)(z-0.6065)}\right)$$



1
30
s^2+7s+10
Ys
.2139
z
z^2-1.4252614z+.4965853
Yz



Plot
Ys
Yz
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0
0
.5
1
1.5
2
2.5
3
3.5
4
Time (sec)

4) $G(s) = \left(\frac{600}{(s^2+2s+15)(s+20)}\right)$

Convert poles to the z plane as $z = e^{sT}$

```
-->s4 = [-1+j*3.74165,-1-j*3.74165,-20]'

  - 1. - 3.74165i
  - 1. + 3.74165i
  - 20.

-->z4 = exp(s4*T)

    0.8422346 - 0.3307139i
    0.8422346 + 0.3307139i
    0.1353353

-->poly(z4)

    1.  - 1.8198044    1.0466989  - 0.1108032
```

$$G(z) \approx \left(\frac{k}{(z-0.8422+j0.3307)(z-0.8422-j0.3307)(z-0.1353)}\right)$$

To find k, match the DC gain

$$\left(\frac{600}{(s^2+2s+15)(s+20)}\right)_{s=0} = 2$$

$$\left(\frac{k}{(z-0.8422+j0.3307)(z-0.8422-j0.3307)(z-0.1353)}\right)_{z=1} = 2$$

k = 0.2321826

This matches the DC gain. To find how many zeros belong, match the pahse at some frequency, like s = j1

$$\left(\frac{600}{(s^2+2s+15)(s+20)}\right)_{s=j1} = 2.1186\angle-10.99^0$$

$$\left(\frac{0.23218}{(z-0.8422+j0.3307)(z-0.8422-j0.3307)(z-0.1353)}\right)_{s=j1} = 0.2121\angle-20.57^0$$

You can fix the phase by adding zeros at z=0. Each zero adds 5.73 degrees

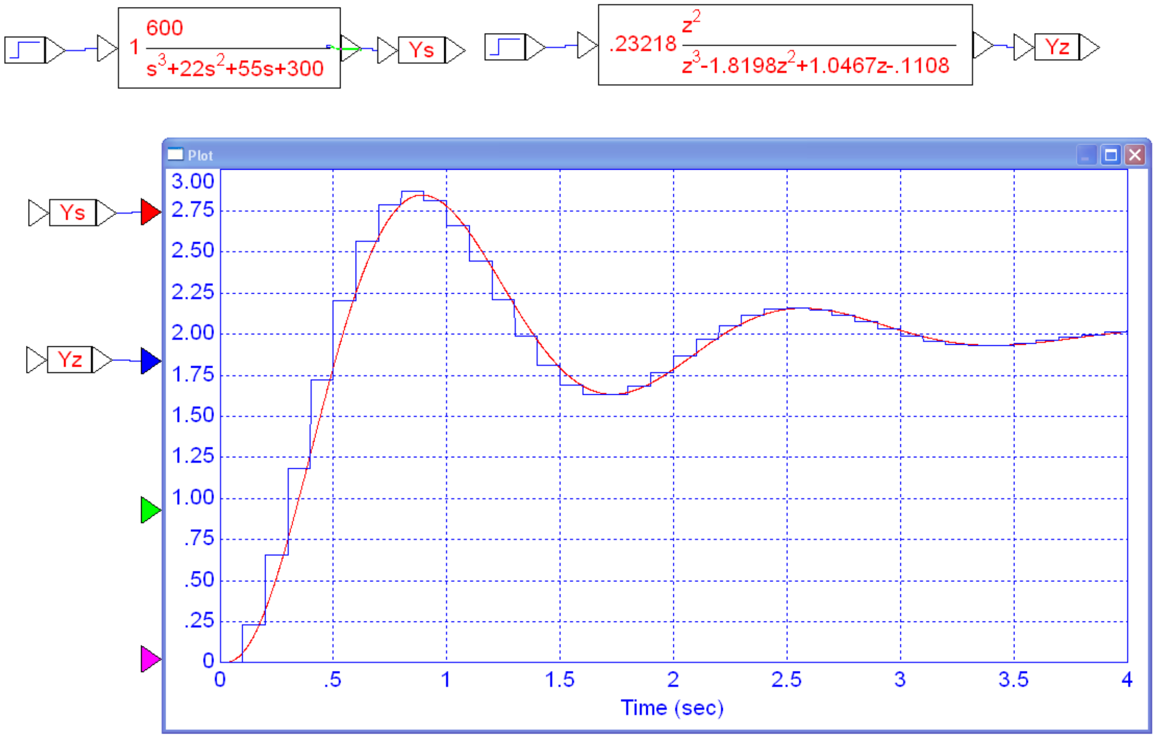
$$(e^{sT})_{s=j1} = 1\angle 5.73^0$$

The numeber of zeros you need to make the phase match is 1.67

$$\left(\frac{20.57-10.99}{5.73}\right) = 1.67$$

Rounding to two zeros at z=0 results in

$$G(z) \approx \left(\frac{0.23218z^2}{(z-0.8422+j0.3307)(z-0.8422-j0.3307)(z-0.1353)}\right)$$



1
600
s3+22s2+55s+300
Ys
.23218
z2
z3-1.8198z2+1.0467z-.1108
Yz
Plot
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0
0
.5
1
1.5
2
2.5
3
3.5
4
Time (sec)

5) $G(s) = \left(\frac{625}{(s+1.31)(s+5.71)(s+12.45)(s+18.37)}\right)$ (heat equation from HW 5)

```
-->s5 = [-1.31,-5.71,-12.45,-19.37]'
 s5  =

  - 1.31
  - 5.71
  - 12.45
  - 19.37

-->z5 = exp(s5*T)
 z5  =

    0.8772178
    0.5649602
    0.2879409
    0.1441357

-->DC = 625 / prod(s5)
 DC  =

    0.3464764

-->k = DC * prod(1 - z5)
 k  =

    0.0112787
```

$$G(z) \approx \left(\frac{0.112787}{(z-0.8772)(z-0.5649)(z-0.2879)(z-0.1441)}\right)$$

To get a slightly better model, match the delay by matching the phase shift at s = j1

$$\left(\frac{625}{(s+1.31)(s+5.71)(s+12.45)(s+18.37)}\right)_{s=j1} = 0.2700\angle -54.84^0$$

Add zeros at z = 0 to make the phase of G(z) match at s = j

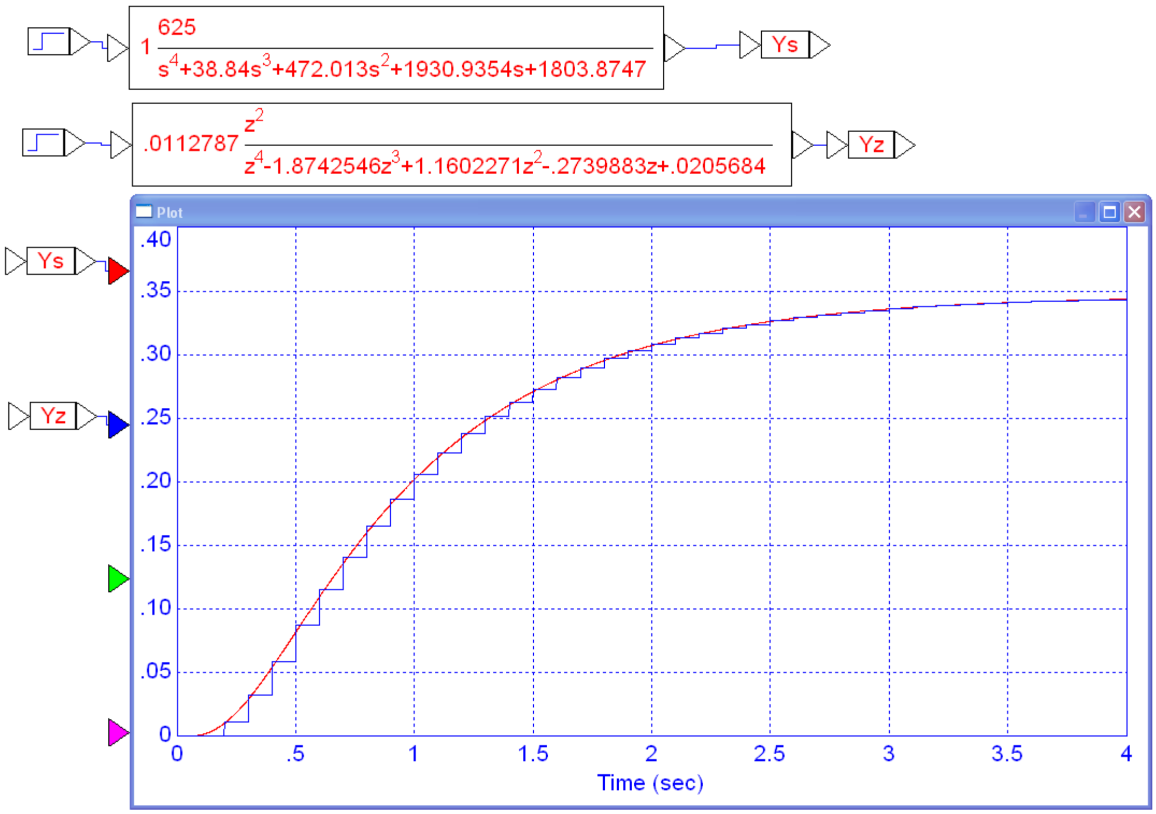
$$\left(\frac{0.112787}{(z-0.8772)(z-0.5649)(z-0.2879)(z-0.1441)}\right)_{s=j} = 0.2704\angle -68.08^0$$

You can fix the phase by adding zeros at z=0. Each zero adds 5.7 degrees. The number of zeros you need is 2.31

$$n = \left(\frac{68.08-54.84}{5.73}\right) = 2.31$$

Rounding to two zeros gives

$$G(z) \approx \left(\frac{0.112787z^2}{(z-0.8772)(z-0.5649)(z-0.2879)(z-0.1441)}\right)$$


625
1
s4+38.84s3+472.013s2+1930.9354s+1803.8747
Ys
.0112787
z2
z4-1.8742546z3+1.1602271z2-.2739883z+.0205684
Yz
Plot
Ys
Yz
.40
.35
.30
.25
.20
.15
.10
.05
0
0
.5
1
1.5
2
2.5
3
3.5
4
Time (sec)

6) $G(s) = \left(\frac{2.5(s+0.25\pm j3.86)}{(s\pm j4.319)(s\pm j1.157)(s+0.25\pm j3.86)}\right)$ (3-mass system from HW 5)

First, convert the poles and zeros to the z-domain. The poles convert as:

```
-->s6 = [j*4.319,-j*4.319,j*1.157,-j*1.157,-0.25+j*3.86,-0.25-j*3.86]'

  - 4.319i
    4.319i
  - 1.157i
    1.157i
  - 0.25 - 3.86i
  - 0.25 + 3.86i

-->z6 = exp(s6*T)

    0.9081721 - 0.4185971i
    0.9081721 + 0.4185971i
    0.9933142 - 0.1154420i
    0.9933142 + 0.1154420i
    0.9035490 - 0.3671903i
    0.9035490 + 0.3671903i
```

The numerator (zeros) convert as:

```
-->s6n = [-0.25+j*3.86,-0.25-j*3.86]'

  - 0.25 - 3.86i
  - 0.25 + 3.86i

-->z6n = exp(s6n*T)

    0.9035490 - 0.3671903i
    0.9035490 + 0.3671903i
```

Matching the DC gain

```
-->DC = 2.5*prod(0 - s6n) / prod(0 - s6)

    0.1001168

-->k = DC*prod(1 - z6) / prod(1 - z6n)

    0.0002459
```

so G(z) is then:

$$G(z) \approx \left(\frac{0.0002459(z-0.9035\pm j0.3671)}{(z-0.9081\pm j0.4185)(z-0.9933\pm j0.1154)(z-0.9035\pm j0.3671)}\right)$$

Add two zeros at z=0 to match the phase (delay) in the step response

In VisSim:

note: you get numerical problems if you multiply out the polynomials. It works better numerically if you cascade second-order systems.