Solution to Homework #4: ECE 461

LaPlace Transforms, 1st and 2nd Order Approximations, Block Diagrams. Due Monday, September 25th

LaPlace Transforms

1) A system has the following transfer function

$$Y = \left(\frac{10(s+3)}{(s+1)(s+4)(s+10)}\right)X$$

1a) What is the differential equation which relates X and Y?

Multiply out the denominator:

```
-->poly([-1,-4,-10])
 ans  =

    1.     15.     54.     40.
```

$$Y = \left(\frac{10s+30}{s^3+15s^2+54s+40}\right)X$$

Cross multiply

$$(s^3+15s^2+54s+40)Y = (10s+30)X$$

'sY' means 'the derivative of Y

$$y''' + 15y'' + 54y' + 40y = 10x' + 30x$$

or

$$\frac{d^3y}{dt^3} + 15\frac{d^2y}{dt^2} + 54\frac{dy}{dt} + 40y = 10\frac{dx}{dt} + 30x$$

1b) Determine y(t) assuming

$$x(t) = 2 + 3\cos(4t)$$

This is a steady-state solution - use phasor analysis.

Use superposition:

x(t) = 2

s = 0

$$\left(\frac{10(s+3)}{(s+1)(s+4)(s+10)}\right)_{s=0} = 0.75$$

$$y = (0.75)(2)$$

$$y = 1.5$$

x(t) = 3 cos(4t)

s = j4

$$\left(\frac{10(s+3)}{(s+1)(s+4)(s+10)}\right)_{s=j4} = 0.199\angle-89^0$$

$$y = (0.199\angle-89^0)\cdot 3\cos(4t)$$

$$y(t) = 0.597\cos(4t-89^0)$$

Add up the two inputs to get the total input. Add up the two outputs to get the total output

$$y(t) = 1.5 + 0.597\cos(4t-89^0)$$

1c) Determine y(t) assuming

$$x(t) = \begin{cases} 0 & t < 0 \\ 2 & t > 0 \end{cases}$$

This is transient analysis: use LaPlace transforms.

The LaPlace transform of x(t) is

$$X(s) = \frac{2}{s}$$

Y(s) is then

$$Y = \left(\frac{10(s+3)}{(s+1)(s+4)(s+10)}\right)\left(\frac{2}{s}\right)$$

To find y(t), use partial fractions

$$Y = \left(\frac{1.5}{s}\right) + \left(\frac{-1.481}{s+1}\right) + \left(\frac{-0.278}{s+4}\right) + \left(\frac{0.2593}{s+10}\right)$$

Now take the inverse LaPlace transform

$$y(t) = 1.5 - 1.481e^{-t} - 0.278e^{-4t} + 0.2593e^{-10t}$$ t > 0

2a) Determine a 2nd-order system which has approximately the same step response as this system

$$Y = \left(\frac{100{,}000}{(s+2)(s+8)(s+20)(s+50)}\right)X$$

Keep the two slowest (most dominant poles).

$$Y \approx \left(\frac{?}{(s+2)(s+8)}\right)X$$

Match the DC gain

$$\left(\frac{100{,}000}{(s+2)(s+8)(s+20)(s+50)}\right)_{s=0} = 6.25$$

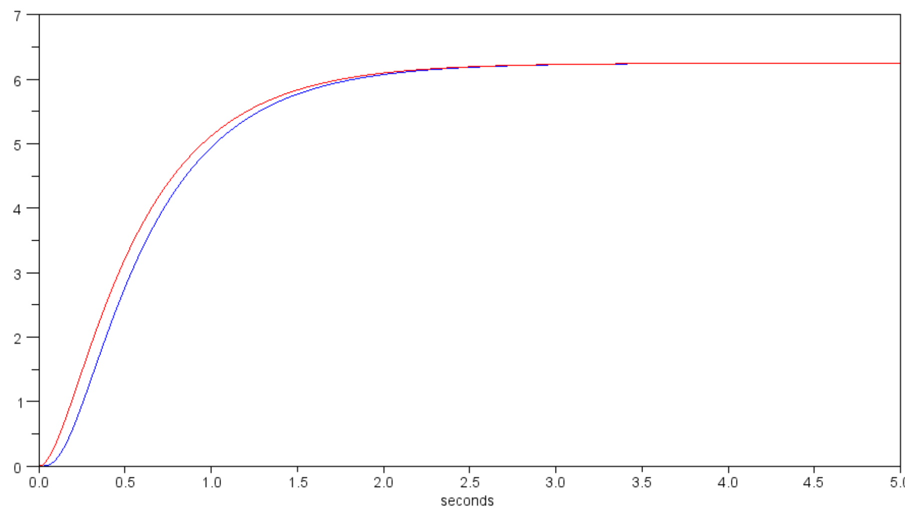
$$\left(\frac{?}{(s+2)(s+8)}\right)_{s=0} = 6.25$$

This gives

$$Y \approx \left(\frac{100}{(s+2)(s+8)}\right)X$$

2b) Compare the step response of the two systems in Matlab (or similar program)

```
-->t = [0:0.01:5]';
-->G4 = zpk([],[-2,-8,-20,-50],100000);
-->G2 = zpk([],[-2,-8],100);
-->y4 = step(G4,t);
-->y2 = step(G2,t);
-->plot(t,y4,t,y2)
-->xlabel('seconds');
```



Step response of the 4th-order system (blue) and 2nd-order approximation (red)
The two systems have the same DC gain, the same 2% settling time, the same overshoot, and the same frequency of oscillation.

3a) Determine a 2nd-order system which has approximately the same step response as this system

$$Y = \left(\frac{100{,}000}{(s^2+2s+16)(s+20)(s+50)} \right) X$$

Keep the two slowest (dominant) poles

$$Y \approx \left(\frac{?}{s^2+2s+16} \right) X$$

Match the DC gain

$$\left(\frac{100{,}000}{(s+2)(s+8)(s+20)(s+50)} \right)_{s=0} = 6.25$$

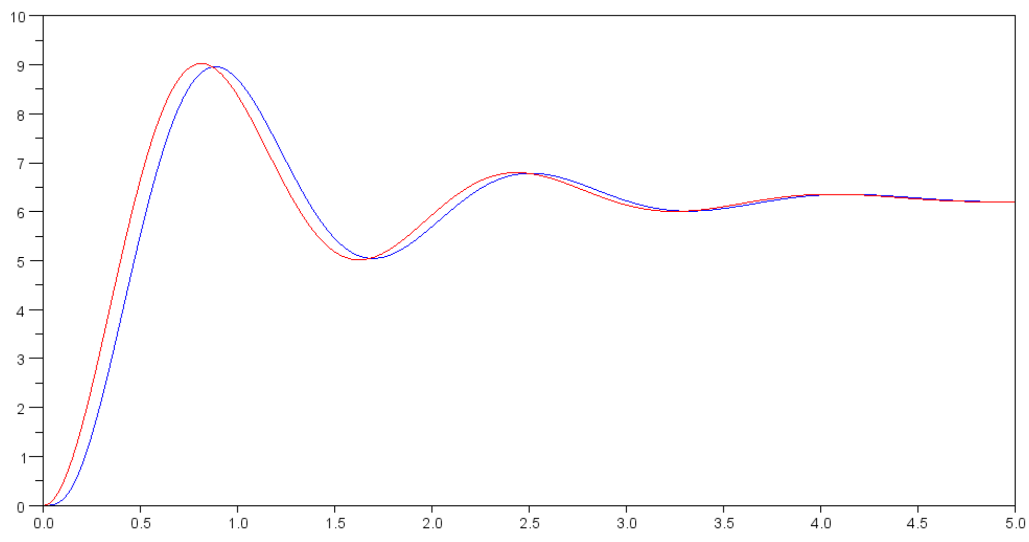
$$\left(\frac{?}{s^2+2s+16} \right)_{s=0} = 6.25$$

This gives

$$Y \approx \left(\frac{100}{s^2+2s+16} \right) X$$

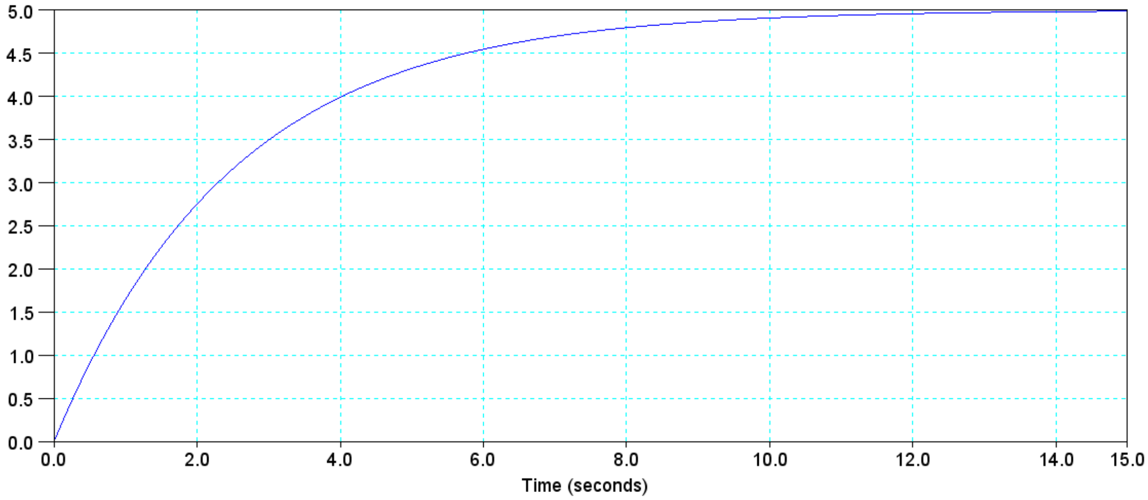
3b) Compare the step response of the two systems in Matlab (or similar program)

```
-->G4 = zpk([],[-1-j*3.873,-1+j*3.873,-20,-50],100000);
-->G2 = tf(100,[1,2,16])
-->y2 = step(G2,t);
-->y4 = step(G4,t);
-->plot(t,y4,t,y2)
```



Step Response of the 4th-order system (blue) and 2nd-order system (red).
The two systems have the same DC gain, the same 2% settling time, the same overshoot, and the same frequency of oscillation.

4) Find the transfer function for a system with the following step response:



This is a 1st-order system, so

$$G(s) \approx \frac{a}{s+b}$$

The DC gain is 5.00

$$\frac{a}{b} = 5$$

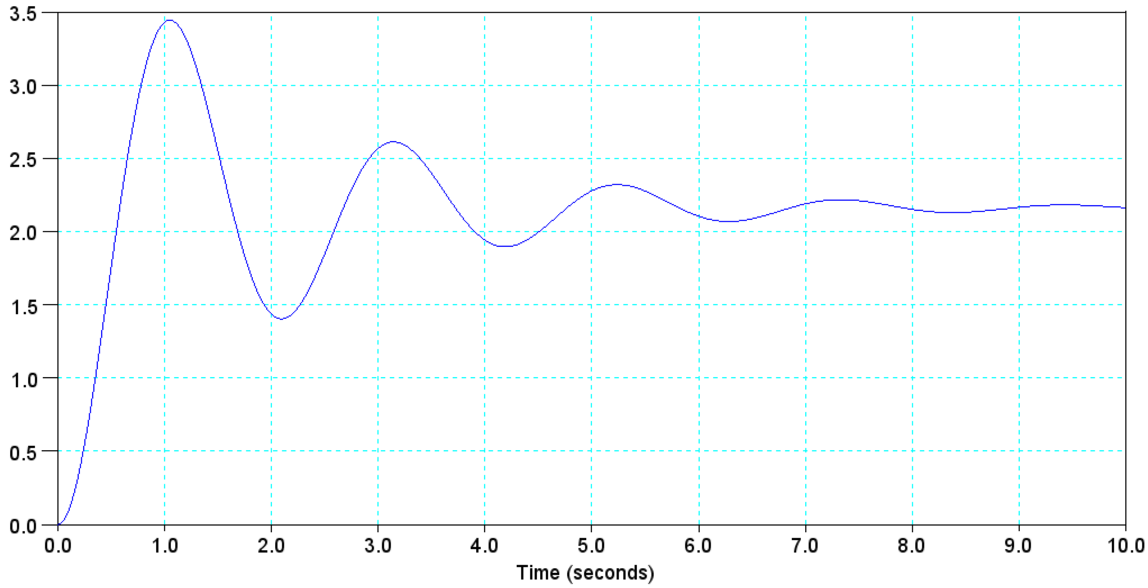
The 2% settling time is 12 seconds (approx)

$$b = \frac{4}{12} = 0.333$$

so

$$G(s) \approx \left(\frac{1.666}{s+0.333}\right)$$

5) Find the transfer function for a system with the following step response:



This is a 2nd-order sytem, so

$$G(s) \approx \left(\frac{a}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)} \right)$$

The DC gain is 2.2

The 2% settling time is 9 seconds (approx)

$$\sigma = \frac{4}{9} = 0.444$$

The frequency of oscillation is

$$\omega_d = \left(\frac{3 \text{ cycles}}{6.2 \text{ sec}} \right) \cdot 2\pi$$

$$\omega_d = 3.04$$

So

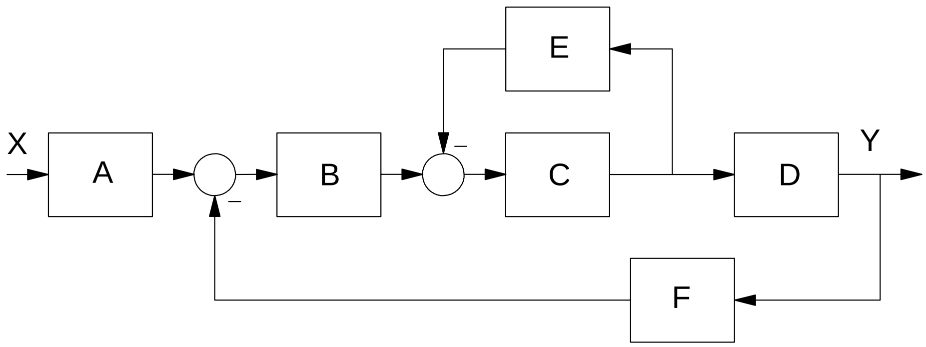
$$G(s) \approx \left(\frac{a}{(s+0.44+j3.04)(s+0.44-j3.04)} \right)$$

Pick the numerator so that the DC gain is 2.2

$$G(s) \approx \left(\frac{20.75}{(s+0.44+j3.04)(s+0.44-j3.04)} \right)$$

Block Diagrams

6) Find the transfer function from X to Y



Shortcut

$$Y=\left(\frac{ABCD}{1+CE+BCDF}\right)X$$

Long way: Combine C and E

$$\left(\frac{C}{1+CE}\right)$$

Combine the outer loop

$$\left(\frac{B\left(\frac{C}{1+CE}\right)D}{1+B\left(\frac{C}{1+CE}\right)DF}\right)$$

Add in A

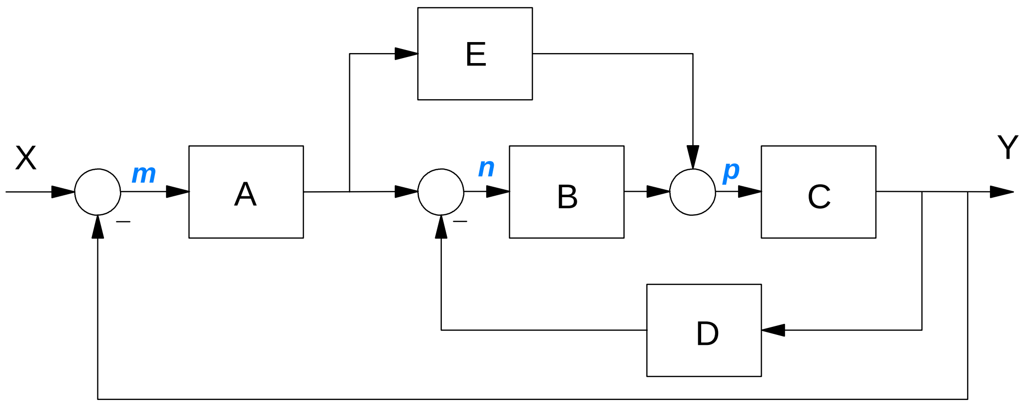
$$Y=A\left(\frac{B\left(\frac{C}{1+CE}\right)D}{1+B\left(\frac{C}{1+CE}\right)DF}\right)X$$

Simplify

$$Y=\left(\frac{ABCD}{(1+CE)+BCDF}\right)X$$

which is the same as the shortcut

7) Find the transfer funciton from X to Y



Shortcut

$$Y = \left(\frac{ABC+AEC}{1+BCD+ABC+AEC} \right) X$$

Long Way

Y = Cp

p = Bn + EAm

n = -DY + Am

m = X - Y

Solve

n = -DY + A (X - Y)

p = B (-DY + A (X - Y)) + EA (X - Y)

Y = Cp = C { B (-DY + A (X - Y)) + EA (X - Y) }

group terms

Y = (-CBD - CBA - EA) Y + (CBA + EA) X

(1 + CBD + CBA + CEA)Y = (CBA + CEA) X

$$Y = \left(\frac{ABC+AEC}{1+BCD+ABC+AEC} \right) X$$