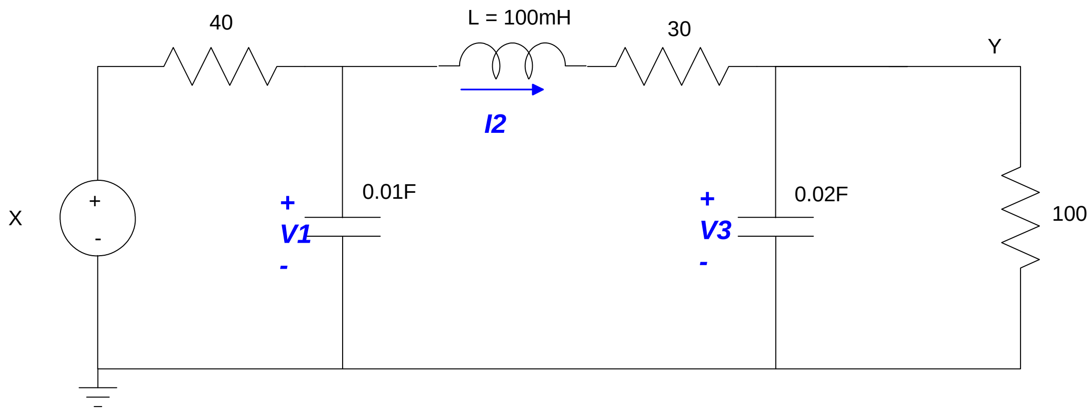
Solution to Homework #5: ECE 461

State Space, Canonical Forms, Heat Equation. Due Monday, October 2nd

1a) Write the differential equations which describe the following circuit



$$C_1\dot{V}_1 = \left(\frac{X-V_1}{40}\right) - I_2$$

$$L_2\dot{I}_2 = V_1 - (V_3 + 30I_2)$$

$$C_3\dot{V}_3 = I_2 - \frac{V_3}{100}$$

1b) Express these dynamics in state-space form

Plugging in numbers and solving for the highest derivative

$$\dot{V}_1 = 2.5X - 2.5V_1 - 100I_2$$

$$\dot{I}_2 = 10V_1 - 10V_3 - 300I_2$$

$$\dot{V}_3 = 50I_2 - 0.5V_3$$

Place in matrix form

$$\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} -2.5 & -100 & 0 \\ 10 & -300 & -10 \\ 0 & 50 & -0.5 \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 2.5 \\ 0 \\ 0 \end{bmatrix} X$$

$$Y = V_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \\ V_3 \end{bmatrix} + [0]X$$

1c) Find the transfer function from X to Y

```
>> A = [-2.5,-100,0 ; 10,-300,-10 ; 0,50,-0.5]
>> B = [2.5 ; 0 ; 0]
>> C = [0 0 1]
>> D = 0
>> G = ss(A,B,C,D);
```

```
>> tf(G)
             1250
-----------------------------
s^3 + 303 s^2 + 2401 s + 2125

>> zpk(G)
             1250
-----------------------------
(s+294.9) (s+7.104) (s+1.014)
```

1d) From the dominant poles, predict what the step response will be like:

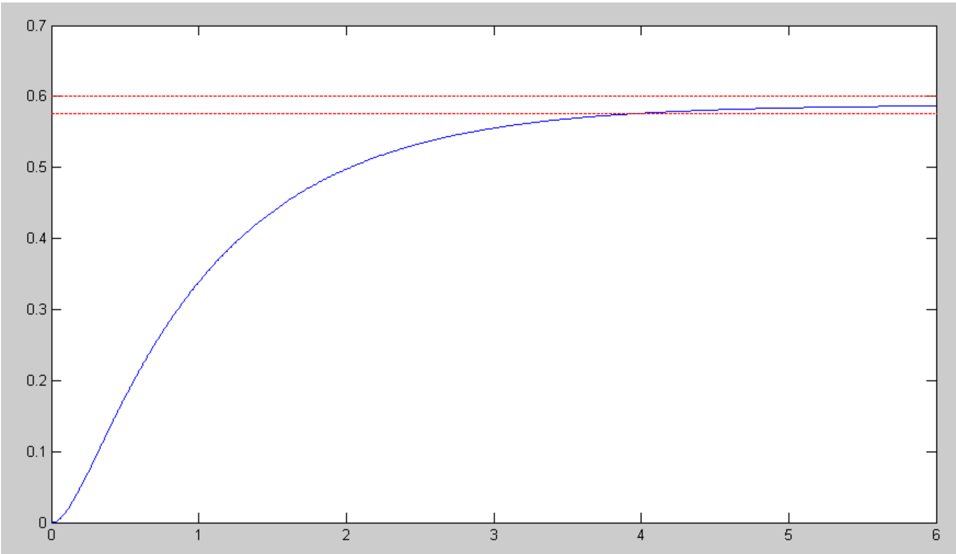
- DC gain
- 2% settling time
- % overshoot for a step input

The dominant pole is at -1.014

- The DC gain is 0.5882
- The 2% settling time is 4 seconds
- There should be no overshoot for a step input.

1e) Find the step response for this system in Matlab and compare the actual response to what you predicted with the 1st or 2nd-order approximation.

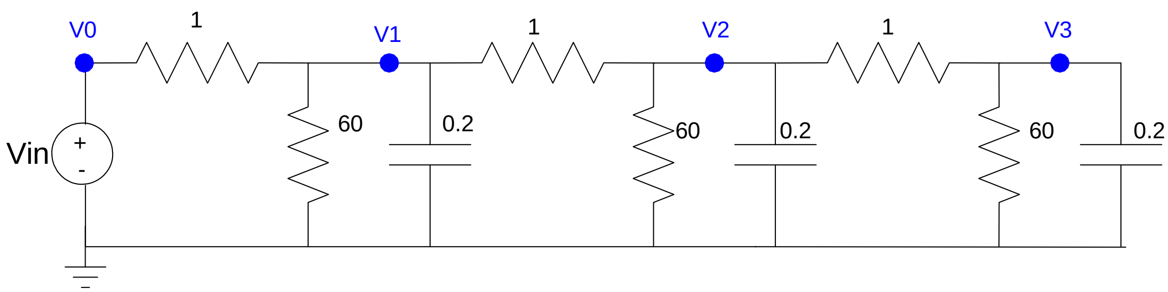
```
>> G = ss(A,B,C,D);
>> DC = evalfr(G,0)

    0.5882

>> t = [0:0.01:6]';
>> y = step(G,t);
>> plot(t,y,'b', t,0.98*DC,'r', t,1.02*DC,'r')
```



	Expected	Actual	Error
DC Gain	0.5882	0.5882	0
Ts	3.95s	4.00s	+1.2%
% Overshoot	none	none	0

3-Stage RC Filter

2a) Write the differential equations which describe the following circuit



$$C\dot{V}_1 = \left(\frac{V_0 - V_1}{1}\right) + \left(\frac{V_2 - V_1}{1}\right) - \left(\frac{V_1}{60}\right)$$

$$C\dot{V}_2 = \left(\frac{V_1 - V_2}{1}\right) + \left(\frac{V_3 - V_2}{1}\right) - \left(\frac{V_2}{60}\right)$$

$$C\dot{V}_3 = \left(\frac{V_2 - V_3}{1}\right) - \left(\frac{V_3}{60}\right)$$

Plugging in numbers

$$\dot{V}_1 = 5V_0 - 10.0833V_1 + 5V_2$$

$$\dot{V}_2 = 5V_1 - 10.0833V_2 + 5V_3$$

$$\dot{V}_3 = 5V_2 - 5.0833V_3$$

2b) Express these dynamics in state-space form

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} -10.0833 & 5 & 0 \\ 5 & -10.0833 & 5 \\ 0 & 5 & -5.0833 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = V_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + [0]V_{in}$$

2c) Find the transfer function from Vin to V3

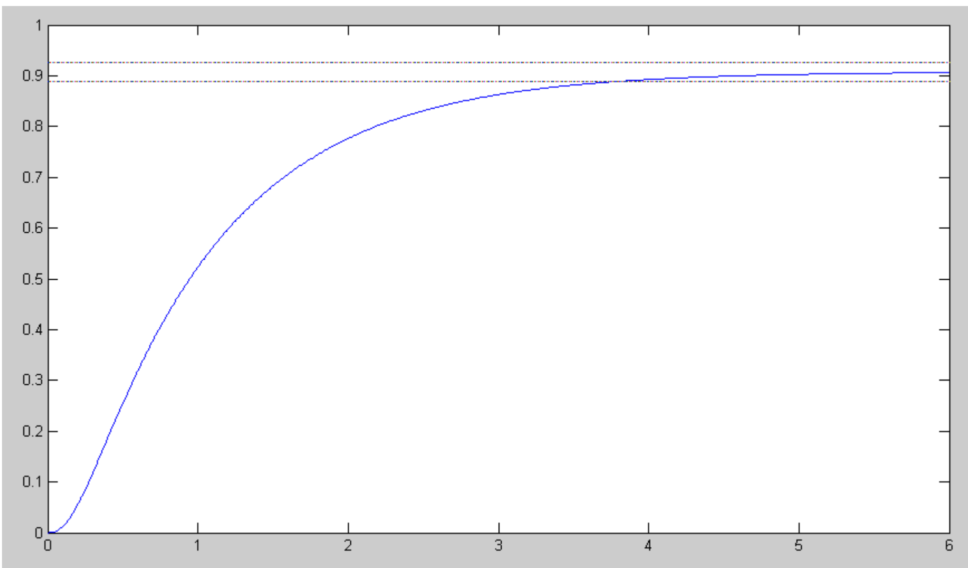
```
>> A = [-10.0833,5,0 ; 5,-10.0833,5 ; 0, 5, -5.0833];
>> B = [5;0;0];
>> C = [0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> tf(G)

                 125
---------------------------------
s^3 + 25.25 s^2 + 154.2 s + 137.7

>> zpk(G)

               125
-----------------------------
(s+16.32) (s+7.858) (s+1.074)
```

2d) From the dominant pole, predict what the step response will be like:

- DC gain = 0.9078
- Ts = 4 seconds (dominant pole is -1.074)
- No overshoot for a step input (real dominant pole)

2e) Find the step response for this system in Matlab and compare the actual response to what you predicted with the 1st-order approximation.

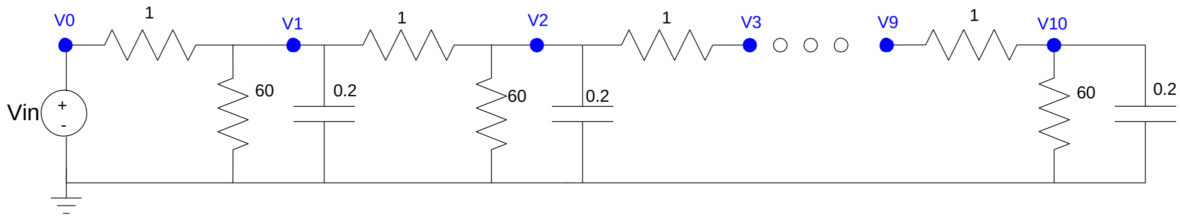
```
>> DC = evalfr(G,0)
>> t = [0:0.01:6]';
>> y = step(G,t);
>> plot(t,y,'b', t,0.98*DC,'r', t,1.02*DC,'r')
```



	Expected	Actual	Error
DC Gain	0.9078	0.9078	0
Ts	3.72s	3.85s	+ 3.5%
% Overshoot	none	none	0

10-Stage RC Filter

3a) Give the state-space model for the 10th-order RC filter shown below (Matlab printout of A,B,C,D preferred)



Same as problem #2. For nodes 1..9

$$\dot{V}_1 = 5V_0 - 10.0833V_1 + 5V_2$$

For node 10

$$\dot{V}_{10} = 5V_9 - 5.0833V_{10}$$

Place in matrix form:

$$s\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix} = \begin{bmatrix} -10.08 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -10.08 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -10.08 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10.08 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -10.08 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -10.08 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -10.08 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -10.08 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -10.08 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5.08 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{V} + [0]V_{in}$$

3b) Find the transfer function from Vin to V10

```
>> A = zeros(10,10);
>> for i=1:9
   A(i,i) = -10.0833;
   A(i+1,i) = 5;
   A(i,i+1) = 5;
   end
>> A(10,10) = -5.0833;

 -10.0833    5.0000         0         0         0         0         0         0         0         0
   5.0000  -10.0833    5.0000         0         0         0         0         0         0         0
        0    5.0000  -10.0833    5.0000         0         0         0         0         0         0
        0         0    5.0000  -10.0833    5.0000         0         0         0         0         0
        0         0         0    5.0000  -10.0833    5.0000         0         0         0         0
        0         0         0         0    5.0000  -10.0833    5.0000         0         0         0
        0         0         0         0         0    5.0000  -10.0833    5.0000         0         0
        0         0         0         0         0         0    5.0000  -10.0833    5.0000         0
        0         0         0         0         0         0         0    5.0000  -10.0833    5.0000
        0         0         0         0         0         0         0         0    5.0000   -5.0833

>> B = [5;0;0;0;0;0;0;0;0;0];
>> C = [0,0,0,0,0,0,0,0,0,1];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                                              9765625
------------------------------------------------------------------------------------------
(s+19.64) (s+18.35) (s+16.32) (s+13.74) (s+10.83) (s+7.858) (s+5.083) (s+2.753) (s+1.074) (s+0.195)

>>
```

3c) From the dominant pole, predict what the step response will be like:

The dominant pole is at -0.195

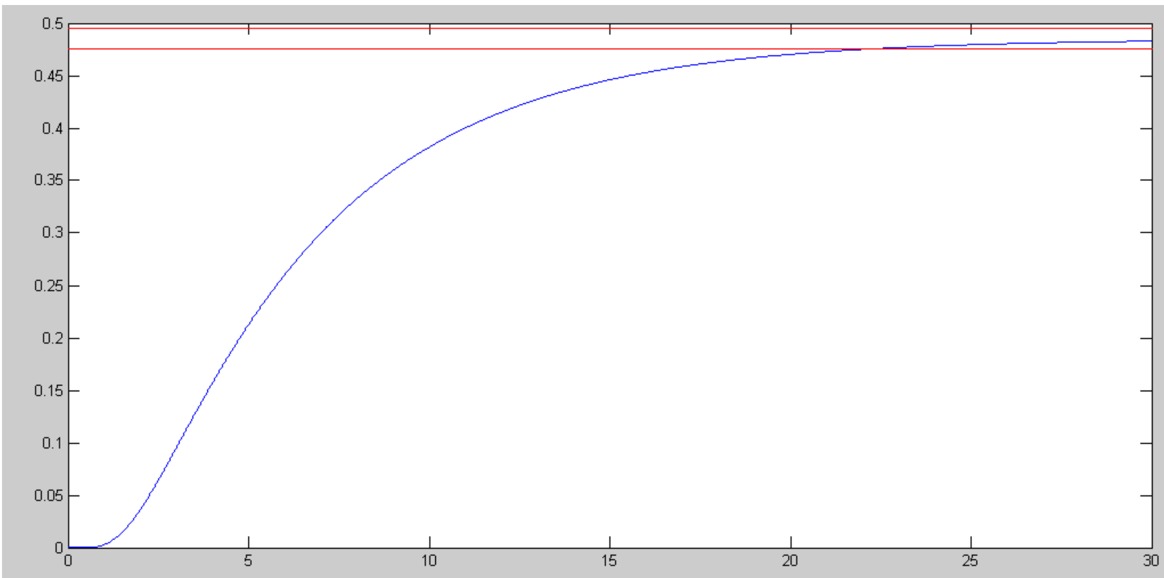
- There should be no oveshoot for a step input (real pole)
- The 2% settling time should be 20.51 seconds

The DC gain is 0.4850

```
>> DC = evalfr(G,0)

    0.4850
```

3d) Find the step response for the 10th-order system in Matlab and compare the actual response to what you predicted with the 1st-order approximation.

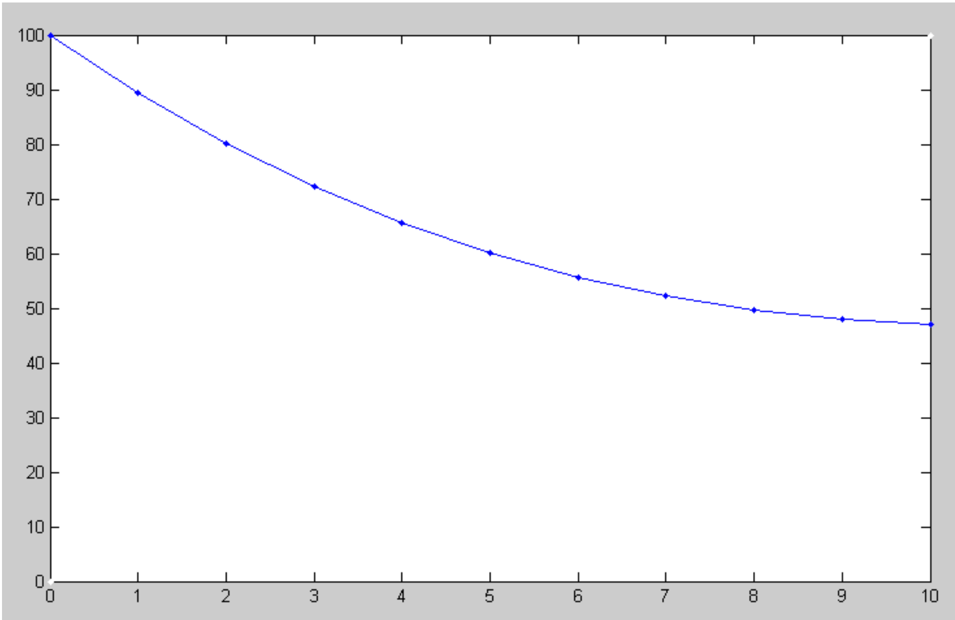
```
>> t = [0:0.01:30]';
>> y = step(G,t);
>> plot(t,y,'b', t,0.98*DC,'r', t,1.02*DC,'r')
```



	Expected	Actual	Error
DC Gain	0.4850	0.4850	0
Ts	20.51s	22.1s	+ 7.7%
% Overshoot	none	none	0

4) Modify the program Heat.m to simulate the above 10th-order RC filter. Give a printout of the resulting code

```
% 10-stage RC Filter

V = zeros(10,1);
dV = zeros(10,1);
V0 = 100;
dt = 0.01;
t = 0;

while(t < 100)

   dV(1)  = 5*V0    - 10.0833*V(1) + 5*V(2);
   dV(2)  = 5*V(1) - 10.0833*V(2) + 5*V(3);
   dV(3)  = 5*V(2) - 10.0833*V(3) + 5*V(4);
   dV(4)  = 5*V(3) - 10.0833*V(4) + 5*V(5);
   dV(5)  = 5*V(4) - 10.0833*V(5) + 5*V(6);
   dV(6)  = 5*V(5) - 10.0833*V(6) + 5*V(7);
   dV(7)  = 5*V(6) - 10.0833*V(7) + 5*V(8);
   dV(8)  = 5*V(7) - 10.0833*V(8) + 5*V(9);
   dV(9)  = 5*V(8) - 10.0833*V(9) + 5*V(10);
   dV(10) = 5*V(9) - 5.0833*V(10);

   V = V +  dV*dt;
   t = t + dt;

   hold off
   plot([0,10],[0,100],'w.');
   hold on
   plot([0:10], [V0;V], '.-');
   pause(0.01);
   end
```



Simulated Temperature Along the Bar after 20 seconds