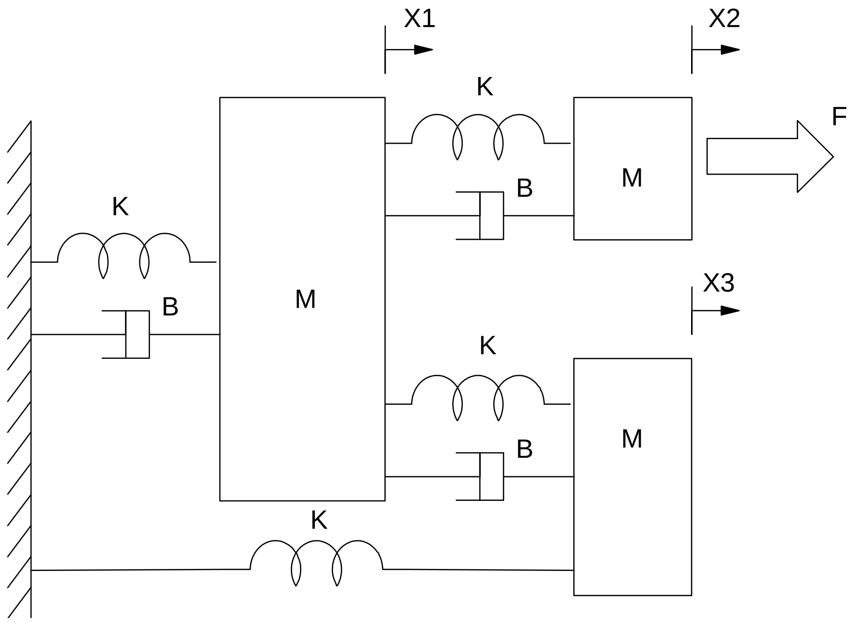
Homework #6: ECE 461

Mass Spring Systems, Rotational Systems, Error Constants. Due Monday, October 9th

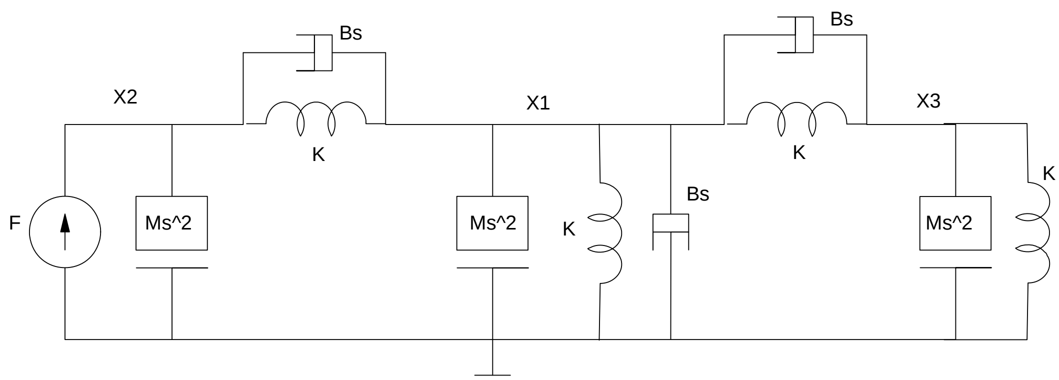
Mass Spring Systems

1) For the following mass-spring system



Problem 1: M = 2kg, K = 5 N/m, B = 0.1 Ns/m

1a) Draw the circuit equivalent



1b) Place this system in state-space form

The voltage node equations are:

$$(Ms^2 + 3Bs + 3K)X_1 - (Bs + K)X_2 - (Bs + K)X_3 = 0$$

$$(Ms^2 + Bs + K)X_2 - (Bs + K)X_1 = F$$

$$(Ms^2 + Bs + 2K)X_3 - (Bs + K)X_1 = 0$$

Solving for the highest derivative

$$Ms^2X_1 = -(3Bs+3K)X_1 + (Bs+K)X_2 + (Bs+K)X_3$$
$$Ms^2X_2 = -(Bs+K)X_2 + (Bs+K)X_1 + F$$
$$Ms^2X_3 = -(Bs+2K)X_3 + (Bs+K)X_1$$

Place in state-space form

$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ \frac{-3K}{M} & \frac{K}{M} & \frac{K}{M} & \vdots & \frac{-3B}{M} & \frac{B}{M} & \frac{B}{M} \\ \frac{K}{M} & \frac{-K}{M} & 0 & \vdots & \frac{B}{M} & \frac{-B}{M} & 0 \\ \frac{K}{M} & 0 & \frac{-2K}{M} & \vdots & \frac{B}{M} & 0 & \frac{-B}{M} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 0 \end{bmatrix}\overline{X}$$

1c) Find the transfer function from F to X1

```
>> K = 5;
>> M = 2;
>> B = 0.1;
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-3*K/M,K/M,K/M ; K/M,-K/M,0 ; K/M,0,-2*K/M];
>> a22 = [-3*B/M,B/M,B/M ; B/M,-B/M,0 ; B/M,0,-B/M];
>> A = [a11,a12 ; a21,a22]

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -7.5000    2.5000    2.5000   -0.1500    0.0500    0.0500
    2.5000   -2.5000         0    0.0500   -0.0500         0
    2.5000         0   -5.0000    0.0500         0   -0.0500

>> B = [0;0;0;0;1/M;0]

         0
         0
         0
         0
    0.5000
         0

>> C = [1,0,0,0,0,0];
>> G = ss(A,B,C,0);
>> zpk(G)

                    0.025 (s+50) (s^2 + 0.05s + 5)
-----------------------------------------------------------------------
(s^2 + 0.0191s + 1.17) (s^2 + 0.04701s + 4.132) (s^2 + 0.1839s + 9.698)
```

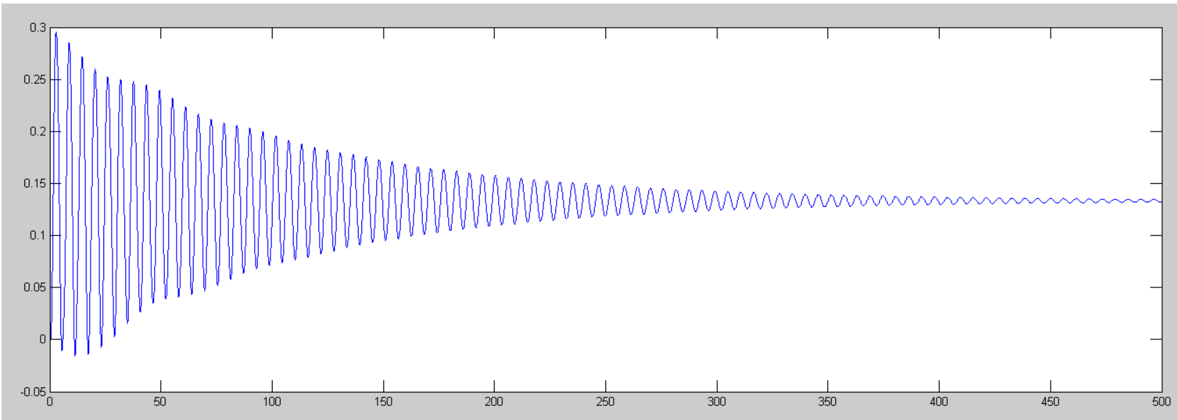
1d) Find the step response from F to X1

Determine the 2% settling time so you know how long to plot the step response:

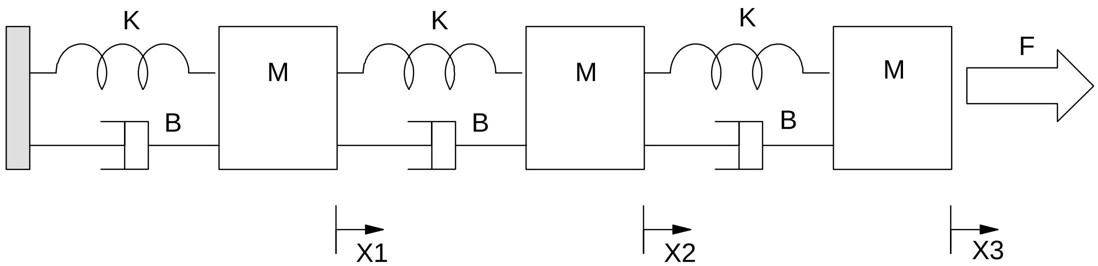
```
>> eig(A)

  -0.0919 + 3.1128i
  -0.0919 - 3.1128i
  -0.0235 + 2.0325i
  -0.0235 - 2.0325i
  -0.0095 + 1.0816i
  -0.0095 - 1.0816i

>> 4/0.0095

  421.0526
```

The 2% settling time is 421 seconds, so plot it out to 500 seconds

```
>> t = [0:0.01:500]';
>> y = step(G,t);
>> plot(t,y)
>>
```

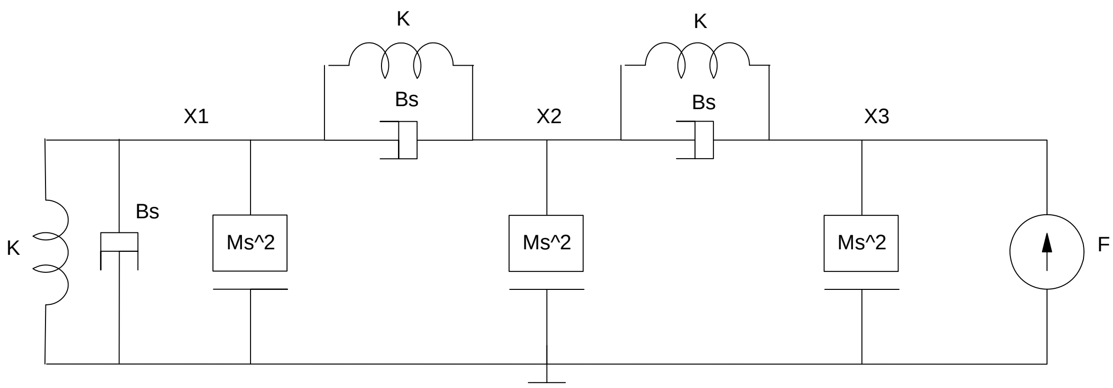


2) For the following mass-spring system



Problem 2: M = 2kg, K = 5 N/m, B = 0.1 Ns/m

2a) Draw the circuit equivalent



2b) Place this system in state-space form

Write the node equations:

$$(Ms^2 + 2Bs + 2K)X_1 - (Bs + K)X_2 = 0$$
$$(Ms^2 + 2Bs + 2K)X_2 - (Bs + K)X_1 - (Bs + K)X_3 = 0$$
$$(Ms^2 + Bs + K)X_3 - (Bs + K)X_2 = F$$

Solve for the highest derivative:

$$Ms^2 X_1 = -(2Bs + 2K)X_1 + (Bs + K)X_2$$
$$Ms^2 X_2 = -(2Bs + 2K)X_2 + (Bs + K)X_1 + (Bs + K)X_3$$
$$Ms^2 X_3 = -(Bs + K)X_3 + (Bs + K)X_2 + F$$

Place this in state-space form

$$s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} = \left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{-2K}{M} & \frac{K}{M} & 0 & \frac{-2B}{M} & \frac{B}{M} & 0 \\ \frac{K}{M} & \frac{-2K}{M} & \frac{K}{M} & \frac{B}{M} & \frac{-2B}{M} & \frac{B}{M} \\ 0 & \frac{K}{M} & \frac{-K}{M} & 0 & \frac{B}{M} & \frac{-B}{M} \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix} F$$

$$Y = \left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \overline{X}$$

2c) Find the transfer function from F to X3

```
>> M = 2;
>> B = 0.1;
>> K = 5;
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-2*K/M,K/M,0 ; K/M,-2*K/M,K/M ; 0,K/M,-K/M];
>> a22 = [-2*B/M,B/M,0 ; B/M,-2*B/M,B/M ; 0,B/M,-B/M];
>> A = [a11,a12 ; a21,a22]

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -5.0000    2.5000         0   -0.1000    0.0500         0
    2.5000   -5.0000    2.5000    0.0500   -0.1000    0.0500
         0    2.5000   -2.5000         0    0.0500   -0.0500

>> B = [0;0;0;0;0;1/M]

         0
         0
         0
         0
         0
    0.5000

>> C = [0,0,1,0,0,0];
>> G = ss(A,B,C,0);
>> zpk(G)


                0.5 (s^2 + 0.05s + 2.5) (s^2 + 0.15s + 7.5)
--------------------------------------------------------------------------
(s^2 + 0.009903s + 0.4952) (s^2 + 0.07775s + 3.887) (s^2 + 0.1623s + 8.117)
```

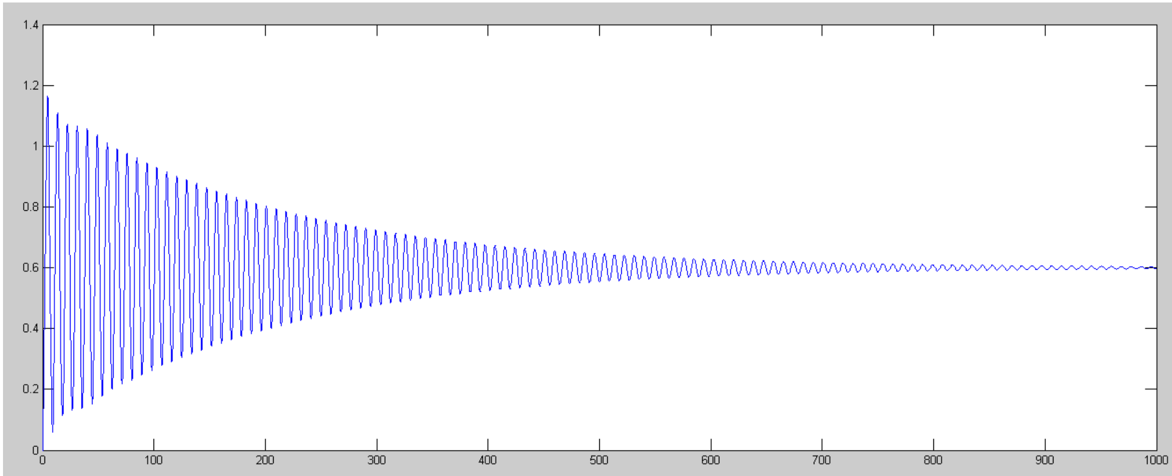
2d) Find the step response from F to X3

Find the 2% settling time to know how long to plot:

```
>> eig(A)

  -0.0812 + 2.8480i
  -0.0812 - 2.8480i
  -0.0389 + 1.9713i
  -0.0389 - 1.9713i
  -0.0050 + 0.7037i
  -0.0050 - 0.7037i

>> 4/0.005

   800
```

Ts = 800s, so plot this out to 1000 seconds

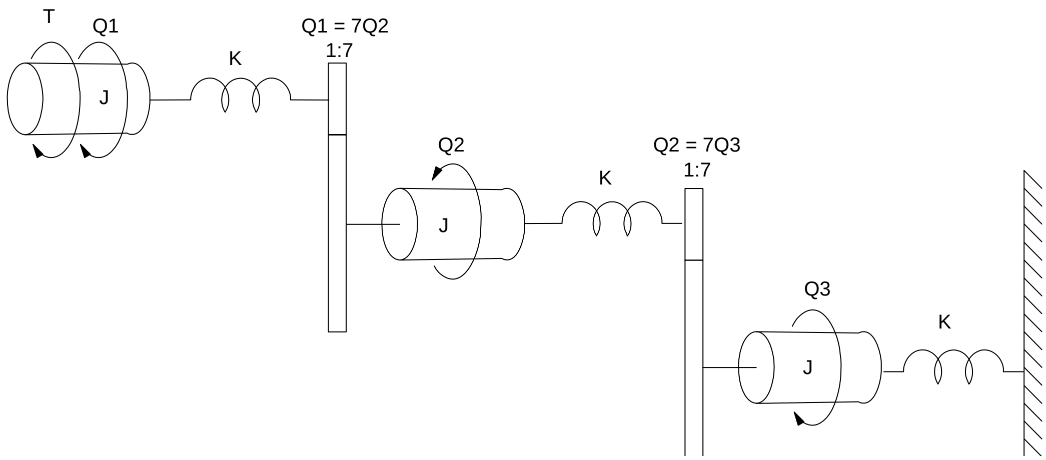
```
>> t = [0:0.01:1000]';
>> y = step(G,t);
>> plot(t,y)
```



1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
100
200
300
400
500
600
700
800
900
1000

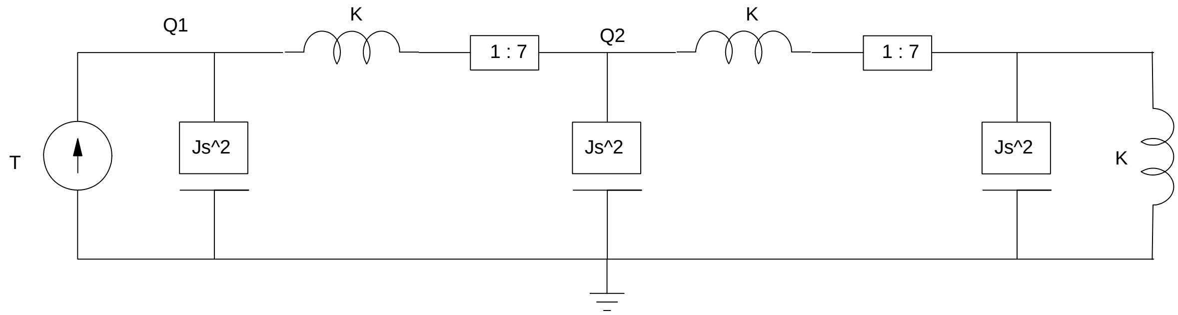
Rotational Systems

3) For the following rotational system



Problem 3: J = 2kg m2, K = 5 N/rad, B = 0.1 Ns/rad

3a) Draw the circuit equivalent



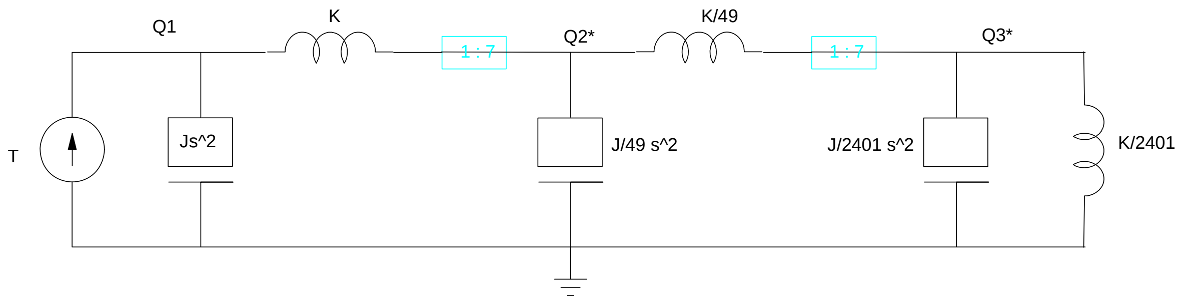
3b) Place this system in state-space form

Remoev the gears. Since part c) goes to Q1, bring everything left. Note that admittances at Q2 are seen at node V1 as

$$Y_1 = \left(\frac{1}{7}\right)^2 Y_2$$

Admittances at node Q3 are seen at node Q1 as

$$Y_1 = \left(\frac{1}{7} \cdot \frac{1}{7}\right)^2 Y_3$$



Now write the voltage node equations

$$(Js^2 + K)\theta_1 - (K)\theta_2 = T$$

$$\left(\frac{J}{49}s^2 + K + \frac{K}{49}\right)\theta_2 - (K)\theta_1 - \left(\frac{K}{49}\right)\theta_3 = 0$$

$$\left(\frac{J}{2401}s^2 + \frac{K}{2401} + \frac{K}{49}\right)\theta_3 - \left(\frac{K}{49}\right)\theta_2 = 0$$

Solve for the highest derivative

$$Js^2\theta_1 = -(K)\theta_1 + (K)\theta_2 + T$$

$$\frac{J}{49}s^2\theta_2 = -\left(K + \frac{K}{49}\right)\theta_2 + (K)\theta_1 + \left(\frac{K}{49}\right)\theta_3$$

$$\frac{J}{2401}s^2\theta_3 = -\left(\frac{K}{2401} + \frac{K}{49}\right)\theta_3 + \left(\frac{K}{49}\right)\theta_2$$

Place in Matrix form

$$s\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \cdots \\ s\theta_1 \\ s\theta_2 \\ s\theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ \frac{-K}{J} & \frac{K}{J} & 0 & \vdots & 0 & 0 & 0 \\ \frac{49K}{J} & \frac{-50K}{J} & \frac{K}{J} & \vdots & 0 & 0 & 0 \\ 0 & \frac{49K}{J} & \frac{-50K}{J} & \vdots & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \cdots \\ s\theta_1 \\ s\theta_2 \\ s\theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} T$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 0 \end{bmatrix}\overline{\theta}$$

3c) Find the transfer function from T to Q1

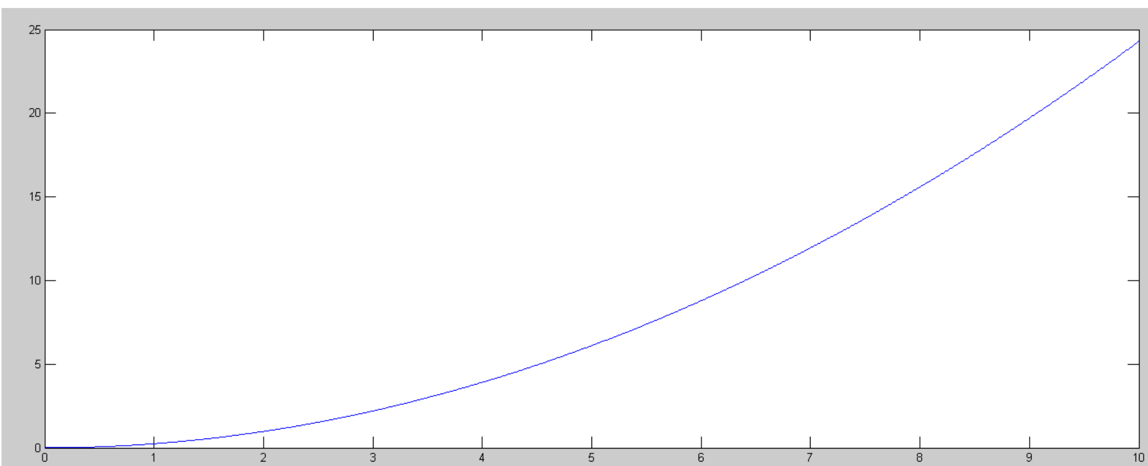
```
>> J = 2;
>> K = 5;
>> B = 0.1;
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-K/J,K/J,0 ; 49*K/J,-50*K/J,K/J ; 0,49*K/J,-50*K/J];
>> a22 = zeros(3,3);
>> A = [a11,a12 ; a21,a22]

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -2.5000    2.5000         0         0         0         0
  122.5000 -125.0000    2.5000         0         0         0
         0  122.5000 -125.0000         0         0         0
```

```
>> B = [0;0;0;1/J;0;0]

         0
         0
         0
    0.5000
         0
         0

>> C = [1,0,0,0,0,0]

     1     0     0     0     0     0

>> G = ss(A,B,C,0);
>> zpk(G)
```

```
       0.5 (s^2 + 107.5) (s^2 + 142.5)
---------------------------------------------
(s^2 + 0.0009992) (s^2 + 108.9) (s^2 + 143.6)

>>
```

3d) Find the step response from T to Q1

```
>> eig(A)

  -0.0000 +11.9841i
  -0.0000 -11.9841i
   0.0000 +10.4346i
   0.0000 -10.4346i
   0.0000 + 0.0316i
   0.0000 - 0.0316i

>> t = [0:0.01:10]';
>> y = step(G,t);
>> plot(t,y)
```



DC Servo Motors

Find the transfer function for the DC servo motors used in the lab. Data on these motors are:

- Ra = 24 Ohms (measured with an ohm-meter)
- La = 12 mH (measured with an inductance meter)

When you apply +10VDC to the motor with no load

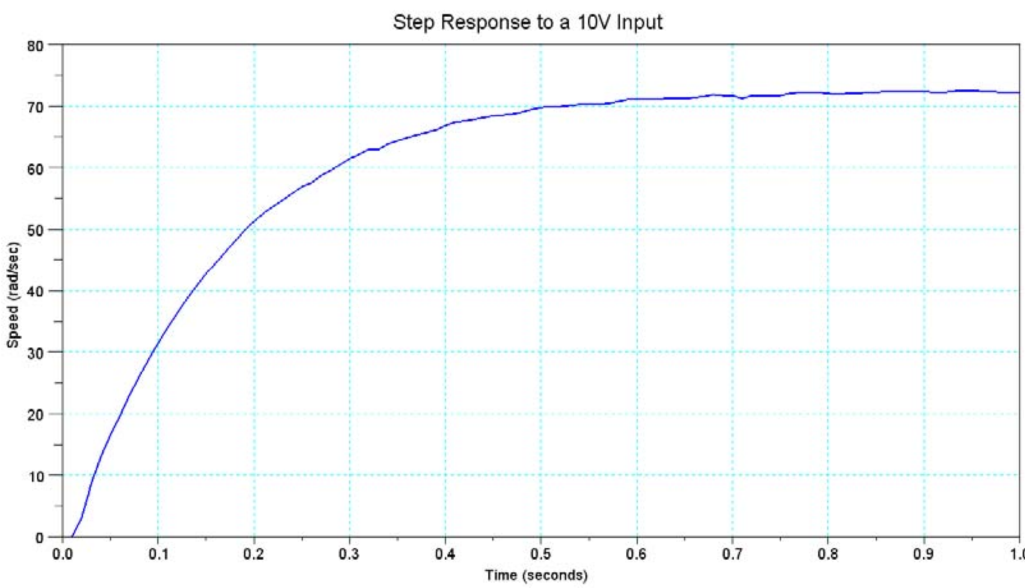
- It spins at 72 rad/sec
- It draws 130mA

$$V = IR + K_t\omega$$

$$10V = (130mA)(24\Omega) + K_t\left(72\tfrac{rad}{\sec}\right)$$

$$K_t = 0.0956\tfrac{V}{rad/\sec} = 0.0956\tfrac{Nm}{A}$$

The step response to a 10VDC step input is as follows (data on-line: 10ms/sample)



This is a 1st-order system, so

$$G(s) = \tfrac{a}{s+b}$$

The 2% settling time is about 0.6 seconds

$$b = \tfrac{4}{0.6} = 6.66$$

The DC gain for a 10V input

$$DC = \frac{72\frac{rad}{\sec}}{10V} = 7.2\tfrac{rad/\sec}{V}$$

$$\tfrac{a}{b} = 7.2$$

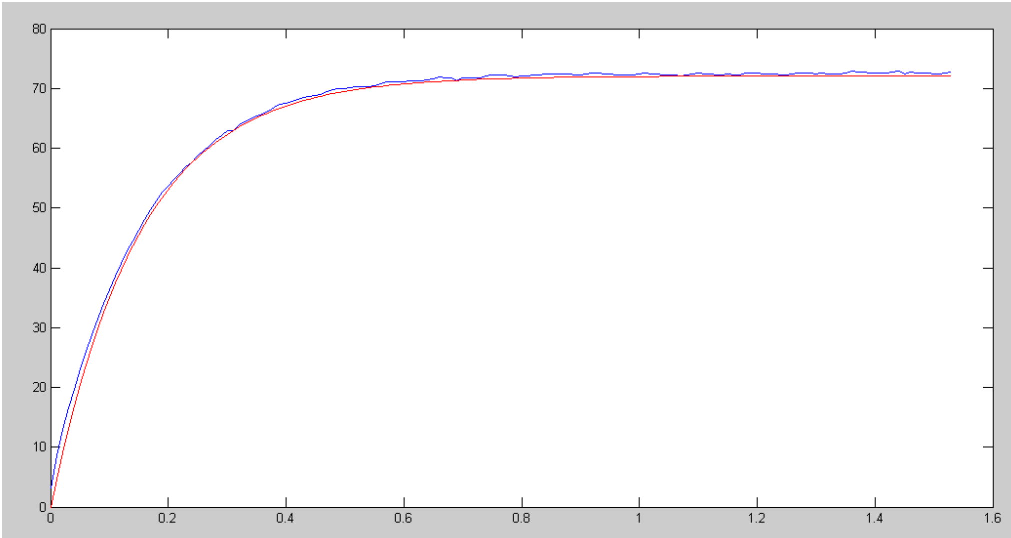
$$a = 48$$

so

$$G(s) = \tfrac{48}{s+6.66}$$

Checking this in matlab:

```
>> G = zpk([],-6.66,48)

Zero/pole/gain:
   48
--------
(s+6.66)

>> wm = step(G,t);
>> plot(t,w,t,wm*10)
```



Step Response to a 10V Input: Measured (blue) and 1st-Order Model (red)

These are pretty close, so this model looks fairly accurate. Matching terms

$$\left(\frac{48}{s+6.66}\right) = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)$$

Assume L = 0

$$\left(\frac{48}{s+6.66}\right) = \left(\frac{K_t}{(Js+D)(R)+K_t^2}\right)$$

$$\left(\frac{48}{s+6.66}\right) = \left(\frac{K_t}{JRs+DR+K_t^2}\right)$$

Putting these in the same form

$$\left(\frac{48}{s+6.66}\right) = \left(\frac{\frac{K_t}{JR}}{s+\frac{DR+K_t^2}{JR}}\right)$$

So

$$\frac{K_t}{JR} = 48 \qquad J = 82.99 \cdot 10^6\ kg\ m^2$$

$$\frac{DR+K_t^2}{JR} = 6.66 \qquad D = 171 \cdot 10^{-6} \frac{Nm}{rad/\sec}$$

Practically, all you care about is the net transfer function:

$$\omega \approx \left(\frac{48}{s+6.66}\right) V$$