Homework #10: ECE 461

z-Transform. Converting G(s) to G(z). Due Monday, November 13th

1) X and Y are related as follows:

$$Y = \left(\frac{400}{(s+10)(s+20)}\right)X$$

a) What is the differential equation relating X and Y?

Cross multiply

$$((s+10)(s+20))Y = (400)X$$

$$(s^2 + 30s + 200)Y = 400X$$

sY means *the derivative of y(t)*

$$\boxed{\frac{d^2y}{dt^2} + 30\frac{dy}{dt} + 200y = 400x}$$

b) Determine y(t) assuming

x(t) = 2 + 3 sin(4t)

Use superposition:

x(t) = 2

s = 0

$$\left(\frac{400}{(s+10)(s+20)}\right)_{s=0} = 2$$

$$y = (2) \cdot 2$$

$$y = 4$$

x(t) = 3 sin(4t)

s = j4

$$\left(\frac{400}{(s+10)(s+20)}\right)_{s=j4} = 1.8209\angle -33.1^0$$

$$y = (1.8209\angle -33.1^0) \cdot 3\sin(4t)$$

$$y = 5.4627\sin(4t - 33.1^0)$$

Putting it all together:

$$\boxed{y = 4 + 5.4627\sin(4t - 33.1^0)}$$

2) X and Y are related as follows:

$$Y = \left(\frac{0.1z}{(z-0.9)(z-0.8)}\right)X$$

a) What is the difference equation relating X and Y?

Cross multiply

$$((z-0.9)(z-0.8))Y = (0.1z)X$$

$$(z^2 - 1.7z + 0.72)Y = (0.1z)X$$

zX means *the next value of X*

y(k+2)-1.7y(k+1)+0.72y(k)=0.1x(k+1)

b) Determine y(t) assuming

x(t) = 2 + 3 sin(4t)

T = 0.1

x(t) = 2

s = 0

$$z = e^{sT} = 1$$

$$\left(\frac{0.1z}{(z-0.9)(z-0.8)}\right)_{z=1} = 5$$

$$y = (5) \cdot 2$$

$$y = 10$$

x(t) = 3 sin(4t)

s = j4

$$z = e^{sT} = 1\angle 22.9^0$$

$$\left(\frac{0.1z}{(z-0.9)(z-0.8)}\right)_{z=1\angle 22.9^0} = 0.6288\angle -136^0$$

$$y = (0.6288\angle -136^0) \cdot 3\sin(4t)$$

$$y = 1.8863\sin(4t - 136^0)$$

Putting it all together:

$$y = 10 + 1.8863\sin(4t - 136^0)$$

3) Assume

$$G(s) = \left(\frac{4}{(s+1)(s+3)}\right)$$

a) Determine a filter, G(z), which has approximately the same step respone as G(s). Assume T = 0.1 sec

s and z are related by

$$z = e^{sT}$$

s = -1: $z = e^{sT} = e^{-0.1} = 0.9048$

s = -3 $z = e^{sT} = e^{-0.3} = 0.7408$

so

$$G(z) = \left(\frac{k}{(z-0.9048)(z-0.7408)}\right)$$

Pick 'k' to match the DC gain

$$\left(\frac{4}{(s+1)(s+3)}\right)_{s=0} = 1.333$$

$$\left(\frac{k}{(z-0.9048)(z-0.7408)}\right)_{z=1} = 1.333$$

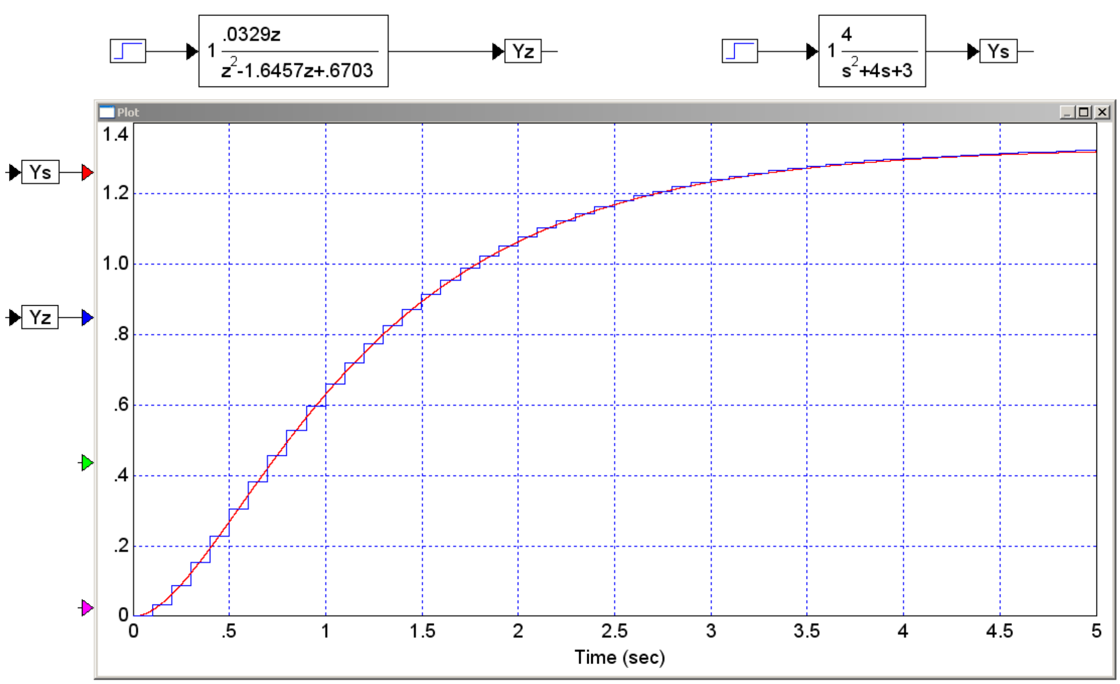
$$k = 0.0329$$

so

$$\boxed{G(z) = \left(\frac{0.0329z}{(z-0.9048)(z-0.7408)}\right)}$$

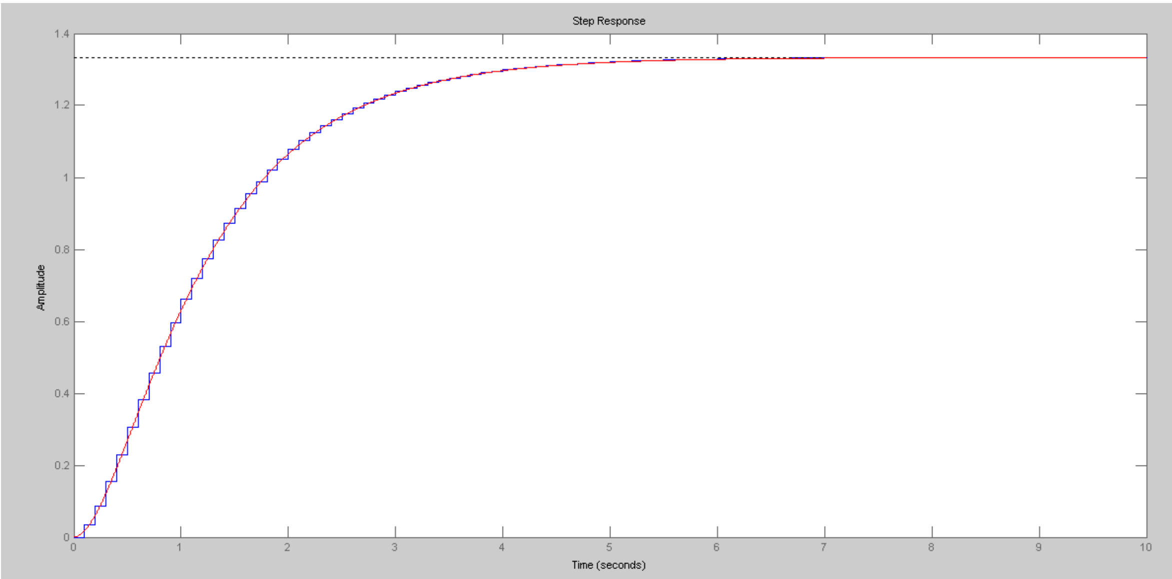
(optional): Add zeros to match the delay. One zero works pretty well.

b) Plot the step response of G(s) and G(z)



In Matlab:

```
>> % The poles in the s-plane
>> ps = [-1,-3]

    -1    -3

>> % The poles in teh z-plane
>> pz = exp(ps*T)

    0.9048    0.7408

>> Gs = zpk([],[-1,-3],4)

     4
-----------
(s+1) (s+3)

>> T = 0.1;
>> Gz = zpk([0],pz,0.0329,T)

      0.0329 z
---------------------
(z-0.9048) (z-0.7408)

Sampling time (seconds): 0.1

>> step(Gz)
>> hold on
>> t = [0:0.001:10]';
>> ys = step(Gs,t);
>> plot(t,ys,'r');
```



4) Assume

$$G(s) = \left(\frac{4}{(s+1+j4)(s+1-j4)}\right)$$

a) Determine a filter, G(z), which has approximately the same step respone as G(s). Assume T = 0.1 sec

s = -1 + j4: $z = e^{sT} = 0.8334 + j0.3524$

so

$$G(z) = \left(\frac{k}{(z-0.8334+j0.3524)(z-0.8334-j0.3524)}\right)$$

Matching the DC gain:

$$\left(\frac{4}{(s+1+j4)(s+1-j4)}\right)_{s=0} = 0.2353$$

$$\left(\frac{k}{(z-0.8334+j0.3524)(z-0.8334-j0.3524)}\right)_{z=1} = 0.2353$$

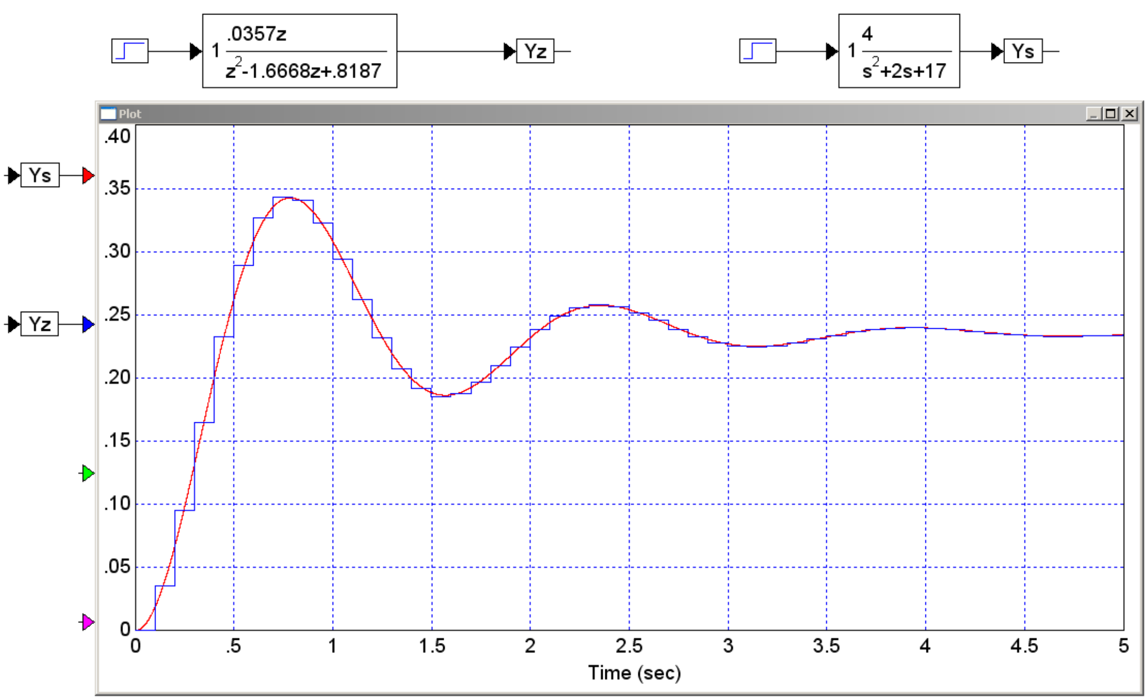
```
k = 0.0357
```

so

$$G(z) = \left(\frac{0.0357z}{(z-0.8334+j0.3524)(z-0.8334-j0.3524)}\right)$$

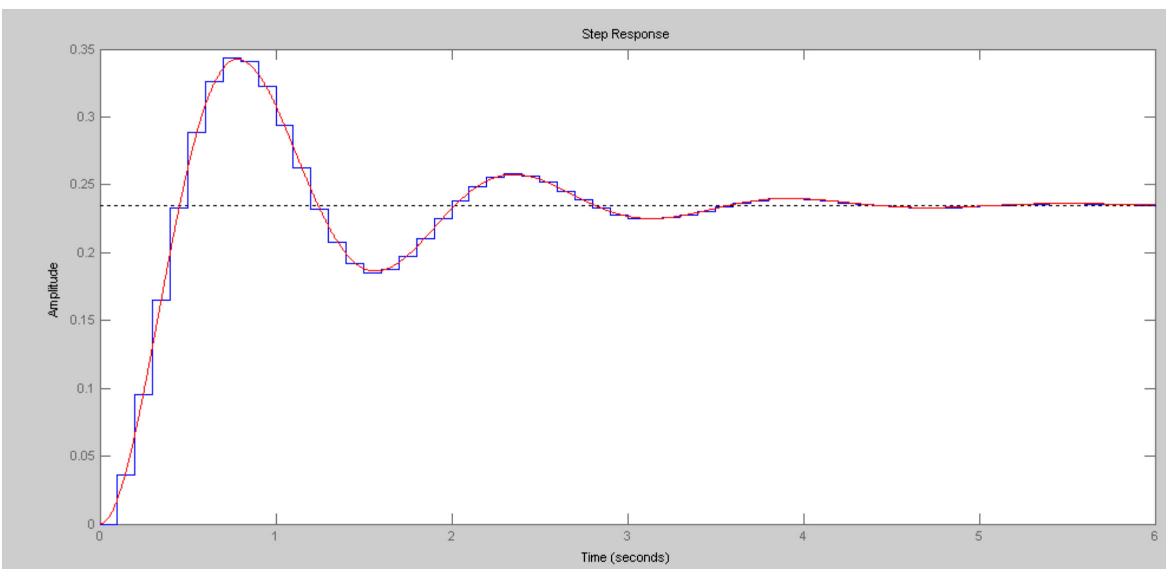
(optional): Add a zero at z = 0 to match the delay. One zero works fairly well.

b) Plot the step response of G(s) and G(z)



In Matlab:

```
>> % The poles in the s-plane
>> ps = [-1+j*4,-1-j*4]

  -1.0000 + 4.0000i  -1.0000 - 4.0000i

>> % The poles in the z-plane
>> pz = exp(ps*T)

   0.8334 + 0.3524i   0.8334 - 0.3524i

>> Gs = zpk([],ps,4)

        4
---------------
(s^2 + 2s + 17)

>> Gz = zpk([0],pz,0.0357,T)

        0.0357 z
-----------------------
(z^2 - 1.667z + 0.8187)

>> step(Gz)
>> hold on
>> t = [0:0.001:6]';
>> ys = step(Gs,t);
>> plot(t,ys,'r');
>> step(Gz)
>> hold on
>> t = [0:0.001:6]';
>> ys = step(Gs,t);
>> plot(t,ys,'r');
```



5) Assume

$$G(s) = 5\left(\frac{s+0.5}{s+2}\right)$$

a) Determine a filter, G(z), which has approximately the same step respone as G(s). Assume T = 0.1 sec

s = -0.5

$$z = e^{sT} = 0.9512$$

s = -2

$$z = e^{sT} = 0.8187$$

so

$$G(z) = k\left(\frac{z-0.9512}{z-0.8187}\right)$$

Pick 'k' to match the DC gain

$$5\left(\frac{s+0.5}{s+2}\right)_{s=0} = 1.25$$

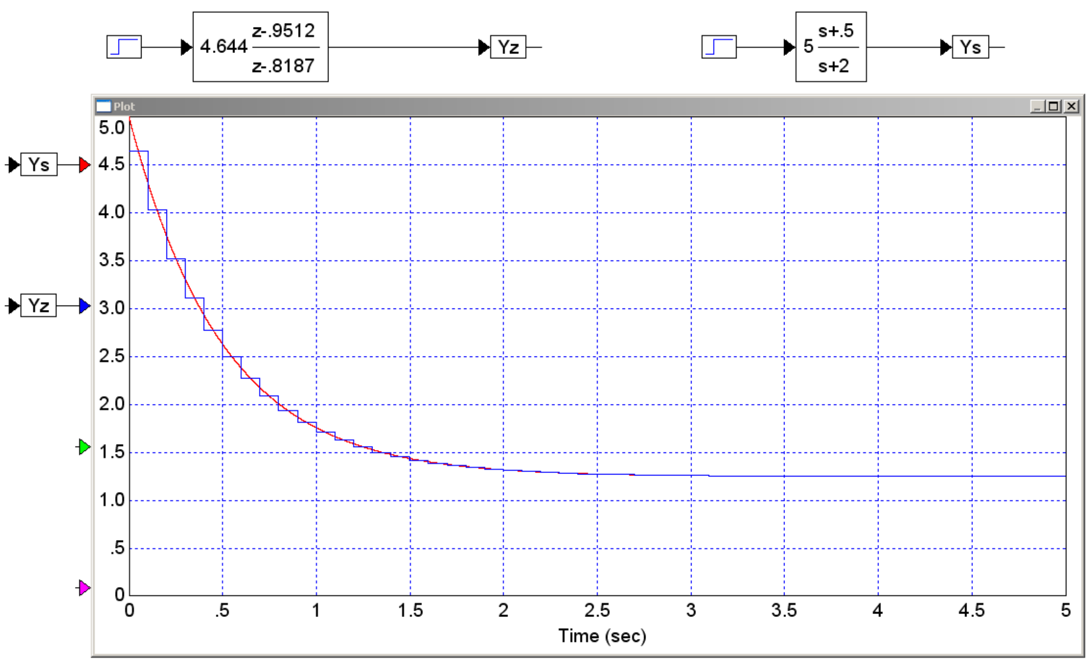
$$k\left(\frac{z-0.9512}{z-0.8187}\right)_{z=1} = 1.25$$

$$k = 4.644$$

so

$$\boxed{G(z) = 4.644\left(\frac{z-0.9512}{z-0.8187}\right)}$$

b) Plot the step response of G(s) and G(z)



In Matlab:

```
>> Gs = zpk(-0.5,-2,5)

5 (s+0.5)
---------
  (s+2)

>> Gz = zpk(exp(-0.5*T),exp(-2*T),4.644,T)

4.644 (z-0.9512)
----------------
   (z-0.8187)

Sampling time (seconds): 0.1

>> step(Gz)
>> hold on
>> t = [0:0.001:3.5]';
>> ys = step(Gs,t);
>> plot(t,ys,'r');
```