Solution #4: ECE 461

LaPlace Transforms - 1st & 2nd-Order Approximations - Block Diagrams

LaPlace Transforms

Problem 1: A system has the following transfer function

$$Y = \left(\frac{10(s+6)}{(s+1)(s+7)(s+10)}\right)X$$

1a) What is the differential equation which relates X and Y?

$$Y = \left(\frac{10(s+6)}{s^3+18s^2+87s+70}\right)X$$

$$(s^3 + 18s^2 + 87s + 70)Y = (10(s+6))X$$

$$\boxed{y''' + 18y'' + 87y' + 70y = 10x' + 60x}$$

1b) Determine y(t) assuming

$$x(t) = 2 + 3\cos(4t)$$

Use superposition

$$x(t) = 2$$

$$s = 0$$

$$\left(\frac{10(s+6)}{(s+1)(s+7)(s+10)}\right)_{s=0} = 0.8571$$

$$y(t) = 0.8571 \cdot 2$$

$$y(t) = 1.7143$$

$$x(t) = 3\cos(4t)$$

$$s = j4$$

$$\left(\frac{10(s+6)}{(s+1)(s+7)(s+10)}\right)_{s=j4} = 0.2014\angle -93.8^0$$

$$y(t) = (0.2014\angle -93.8^0) \cdot 3\cos(4t)$$

$$y(t) = 0.6042\cos(4t - 93.8^0)$$

The total answer is DC + AC

$$\boxed{y(t) = 1.7143 + 0.6042\cos(4t - 93.8^0)}$$

1c) Determine y(t) assuming

$$x(t) = \begin{cases} 0 & t < 0 \\ 2 & t > 0 \end{cases}$$

$$X(s) = \left(\frac{2}{s}\right)$$

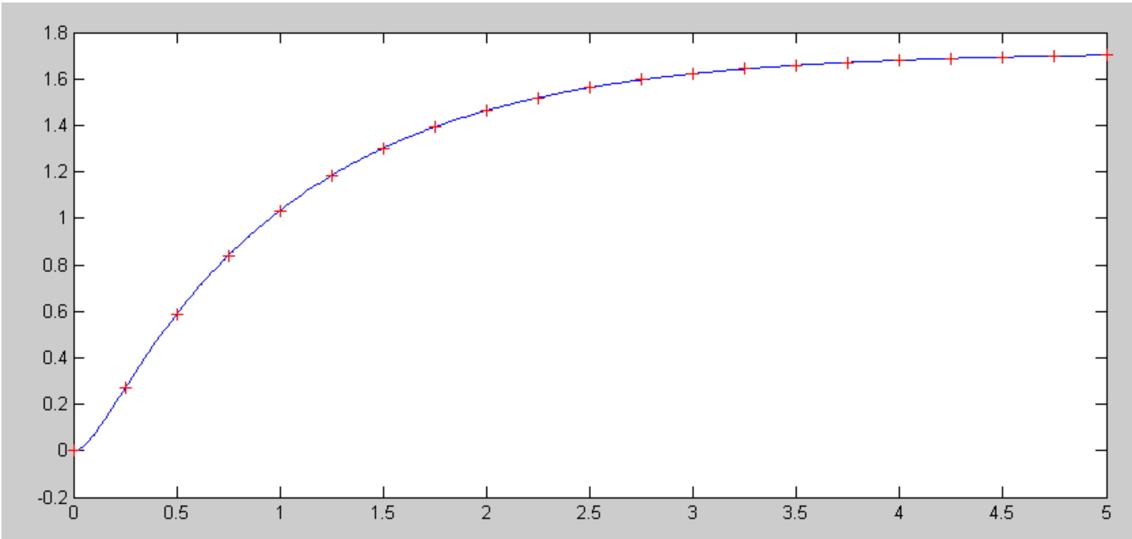
$$Y = \left(\frac{10(s+6)}{(s+1)(s+7)(s+10)}\right)\left(\frac{2}{s}\right)$$

$$Y = \left(\frac{1.7143}{s}\right) + \left(\frac{-1.8519}{s+1}\right) + \left(\frac{-0.1587}{s+7}\right) + \left(\frac{0.2963}{s+10}\right)$$

$$y(t) = (1.7143 - 1.8519e^{-t} - 0.1587e^{-7t} + 0.2963e^{-10t})u(t)$$

1d) Compare your answer for part c) with the response from Matlab

```
t = [0:0.01:5]';
y = 1.7143 - 1.8519*exp(-t) - 0.1587*exp(-7*t) + 0.2963*exp(-10*t);

G = zpk(-6,[-1,-7,-10],10);
y0 = 2 * step(G,t);
N = [1:25:length(t)]';
plot(t,y,t(N),y0(N),'r+')
```



Calculated Step Response (blue) and Matlab Solution (red crosses)

Problem 2: A system has the following transfer function

$$Y = \left(\frac{50}{(s^2+2s+10)(s+30)} \right) X$$

2a) What is the differential equation which relates X and Y?

Multiplication of polynomials is actually convoltion:

```
conv([1,2,10],[1,30])

     1    32    70   300
```

$$Y = \left(\frac{50}{s^3+32s^2+70s+300} \right) X$$

$$(s^3 + 32s^2 + 70s + 300)Y = (50)X$$

$$y''' + 32y'' + 70y' + 300y = 50x$$

2b) Determine y(t) assuming

$$x(t) = 2 + 3\cos(4t)$$

x(t) = 2

$$s = 0$$

$$\left(\frac{50}{(s^2+2s+10)(s+30)} \right)_{s=0} = 0.1667$$

$$y = (0.1667) \cdot 2$$

$$y = 0.3333$$

x(t) = 3 cos(4t)

$$s = j4$$

$$\left(\frac{50}{(s^2+2s+10)(s+30)} \right)_{s=j4} = 0.1652\angle -134^0$$

$$y(t) = (0.1652\angle -134^0) \cdot 3\cos(4t)$$

$$y(t) = 0.4956\cos(4t - 134^0)$$

The total answer includes the DC and AC term

$$y(t) = 0.3333 + 0.4956\cos(4t - 134^0)$$

2c) Determine y(t) assuming

$$x(t) = \begin{cases} 0 & t < 0 \\ 2 & t > 0 \end{cases}$$

$$X(s) = \left(\frac{2}{s}\right)$$

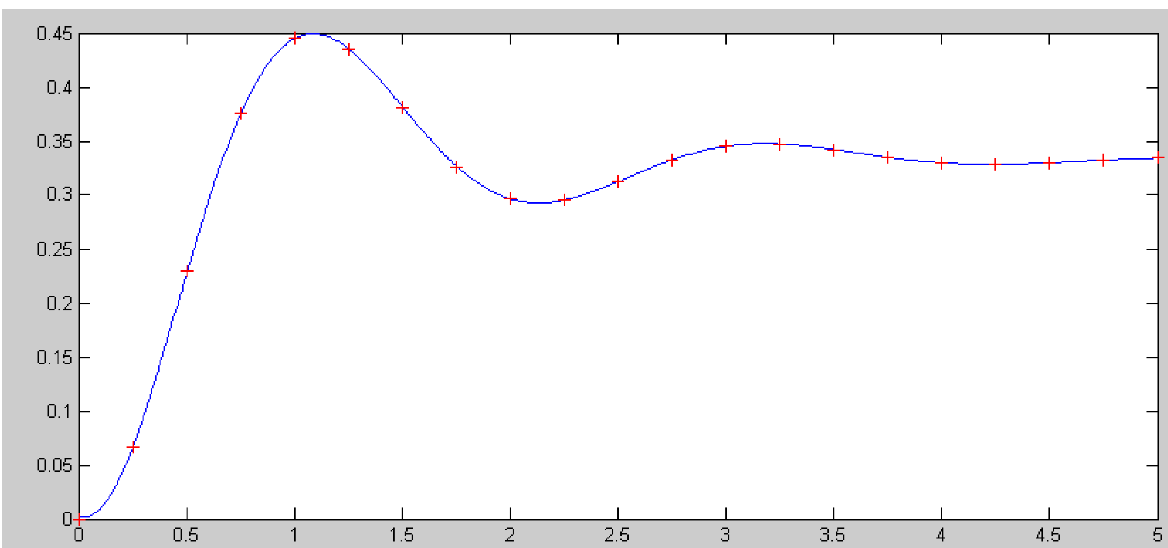
$$Y = \left(\frac{50}{\left(s^2+2s+10\right)(s+30)}\right)\left(\frac{2}{s}\right)$$

$$Y = \left(\frac{0.3333}{s}\right) + \left(\frac{0.1808\angle -155^0}{s+1+j3}\right) + \left(\frac{0.1808\angle 155^0}{s+1-j3}\right) + \left(\frac{-0.0039}{s+30}\right)$$

$$y(t) = (0.3333 + 0.3616e^{-t}\cos(3t + 155^0) - 0.0039e^{-30t})u(t)$$

2d) Compare your answer for part c) with the response from Matlab

```
t = [0:0.01:5]';
y = 0.3333 + 0.3616*exp(-t).*cos(3*t+2.7053) - 0.0039*exp(-30*t);

G = tf(50,[1,32,70,300]);
y0 = 2 * step(G, t);

N = [1:25:length(t)]';
plot(t,y,t(N),y0(N),'r+')
```



Calculated Step Response (blue) and Matlab Solution (red crosses)

1st & 2nd Order Approximations

Problem 3: Determine a 1st-order system which has approximately the same step response as this system

$$Y = \left(\frac{100{,}000}{(s+2)(s+7)(s+10)(s+15)}\right)X$$

The dominant pole is at -2

$$\left(\frac{100{,}000}{(s+2)(s+7)(s+10)(s+15)}\right) \approx \left(\frac{a}{s+2}\right)$$

Matching the DC gain

$$\left(\frac{100{,}000}{(s+2)(s+7)(s+10)(s+15)}\right)_{s=0} = 47.619$$

$$\left(\frac{a}{s+2}\right)_{s=0} = 47.619$$

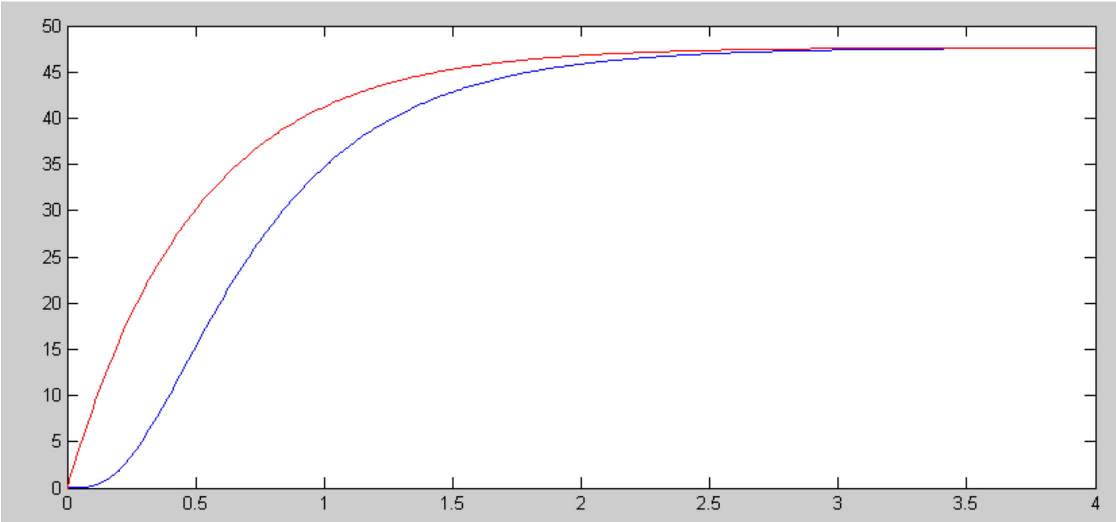
$$a = 95.238$$

so

$$\left(\frac{100{,}000}{(s+2)(s+7)(s+10)(s+15)}\right) \approx \left(\frac{95.318}{s+2}\right)$$

Compare the step response of the two systems in Matlab (or similar program)

```
t = [0:0.01:5]';
G4 = zpk([],[-2,-7,-10,-15],100000);
G1 = tf(95.318,[1,2]);
t = [0:0.01:4]';
y1 = step(G1,t);
y4 = step(G4,t);
plot(t,y4,t,y1,'r');
```



Step Response of 4th-Order System (blue) and 1st-Order Approximation (red)

Problem 4: Determine a 2nd-order system which has approximately the same step response as this system

$$Y = \left(\frac{100{,}000}{(s^2+3s+15)(s+20)(s+50)}\right) X$$

The dominant pole is the complex pair

$$\left(\frac{100{,}000}{(s^2+3s+15)(s+20)(s+50)}\right) \approx \left(\frac{a}{s^2+3s+15}\right)$$

Match the DC gain

$$\left(\frac{100{,}000}{(s^2+3s+15)(s+20)(s+50)}\right)_{s=0} \approx \left(\frac{a}{s^2+3s+15}\right)_{s=0} = 6.6666$$

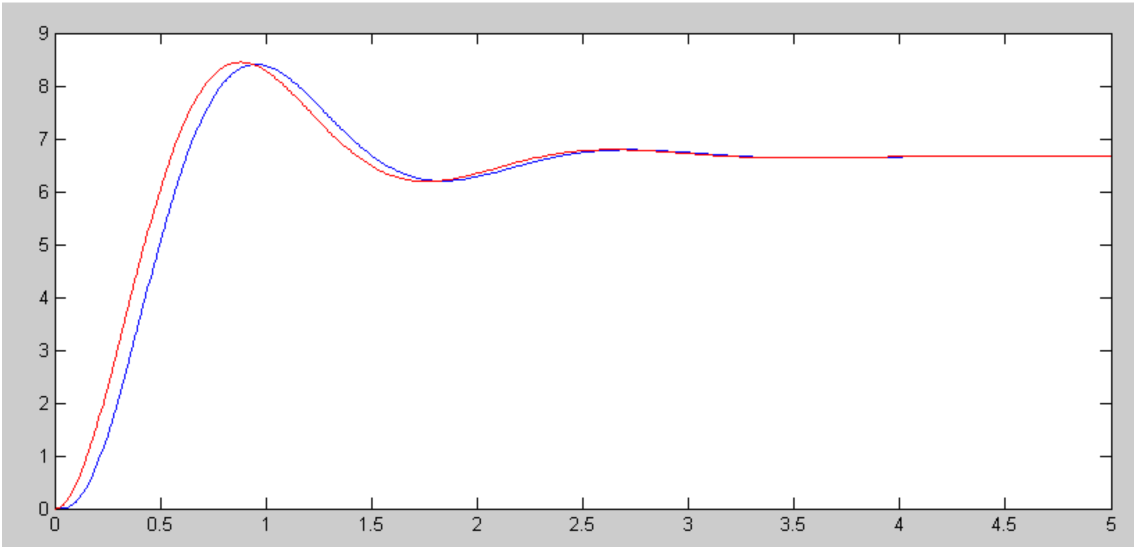
$$a = 100$$

so

$$\left(\frac{100{,}000}{(s^2+3s+15)(s+20)(s+50)}\right) \approx \left(\frac{100}{s^2+3s+15}\right)$$

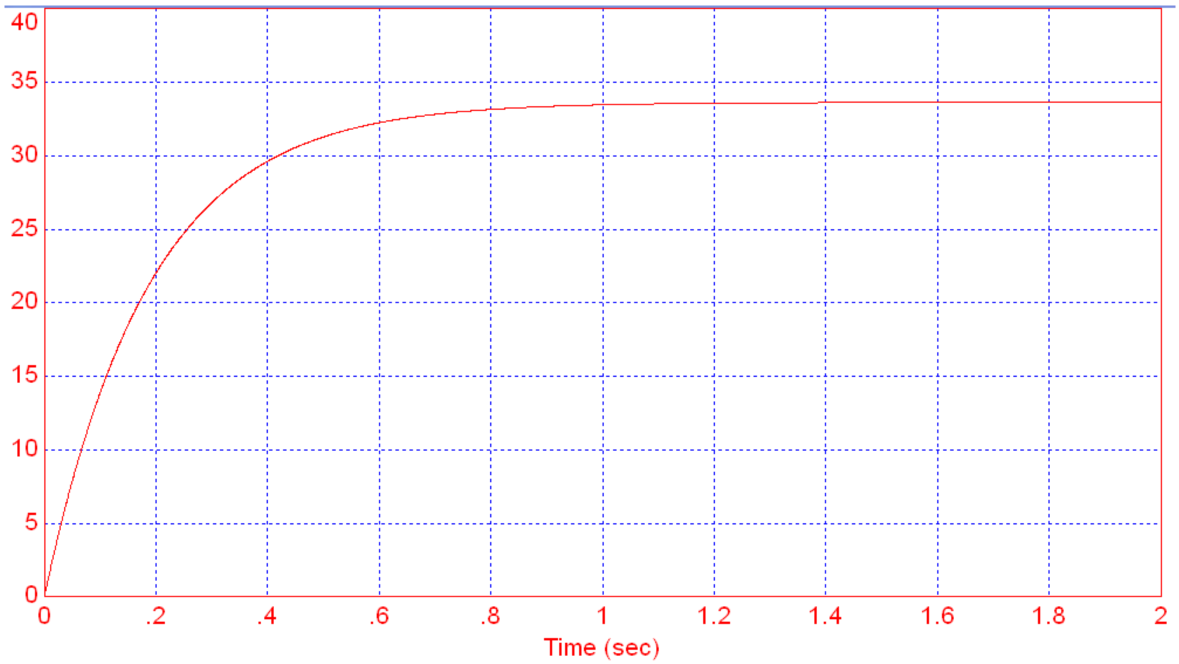
Compare the step response of the two systems in Matlab (or similar program)

```
den = conv([1,3,15],[1,20])
     1    23    75   300

den = conv(den,[1,50])
            1           73         1225         4050        15000

G4 = tf(100000,den);
G1 = tf(100,[1,3,15]);
t = [0:0.01:5]';
y1 = step(G1,t);
y4 = step(G4,t);
plot(t,y4,t,y1,'r');
```



Step Response of 4th-Order System (blue) and 1st-Order Approximation (red)

Problem 5: Find the transfer function for a system with the following step response:



This is a 1st-order system

$$G(s) \approx \left(\frac{a}{s+b}\right)$$

The DC gain is 34

$$\left(\frac{a}{s+b}\right)_{s=0} = \left(\frac{a}{b}\right) = 34$$

The 2% settling time is about 0.8 seconds

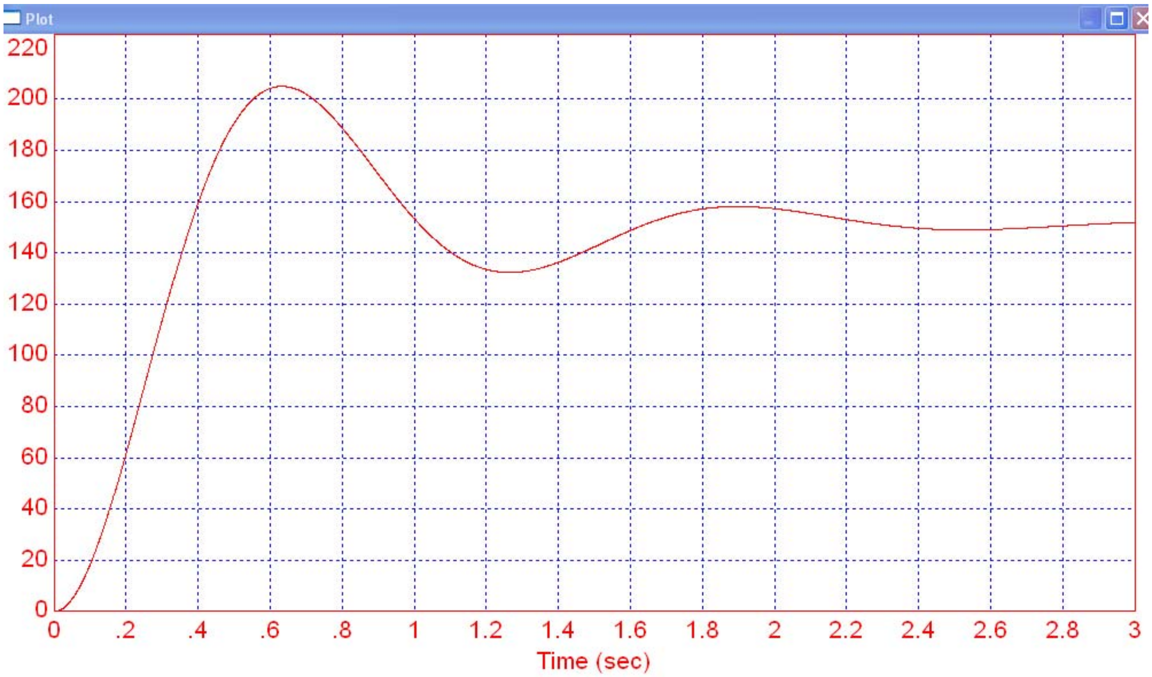
$$t_{2\%} = \left(\frac{4}{b}\right) = 0.8$$

$$b = 5$$

so

$$G(s) \approx \left(\frac{170}{s+5}\right)$$

Problem 6: Find the transfer function for a system with the following step response:



This is a 2nd-order system: It oscillates meaning it has a complex pole along with its conjugate

$$G(s) \approx \left(\frac{a}{(s+b+jc)(s+b-jc)} \right)$$

The DC gain is about 150

$$\left(\frac{a}{(s+b+jc)(s+b-jc)} \right)_{s=0} = 150$$

The 2% settling time is about 2.6 seconds (telling you the real part of the pole)

$$t_{2\%} = \frac{4}{b} = 2.6$$

$$b = 1.5385$$

The frequency of oscillation (tells you the complex part of the pole)

$$c = \left(\frac{2 \text{ cycles}}{2.6 \text{ seconds}} \right) \cdot 2\pi$$

$$c = 4.833$$

so

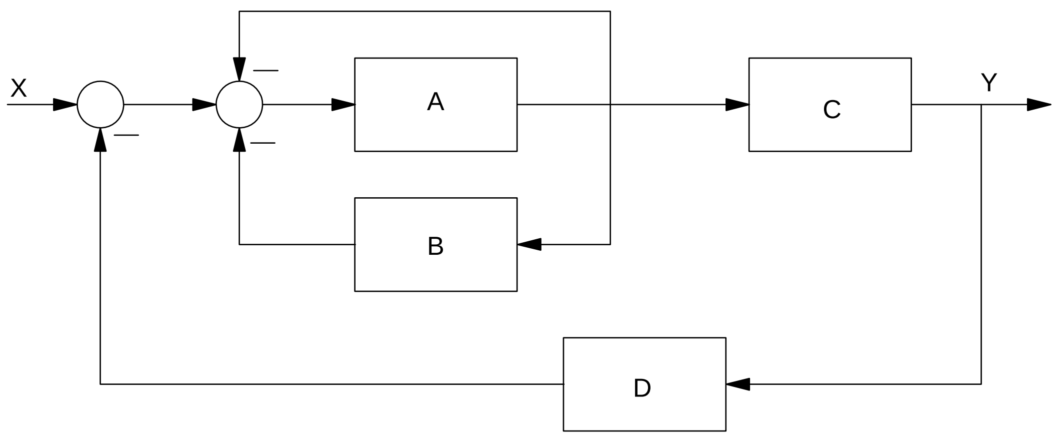
$$G(s) \approx \left(\frac{3859}{(s+1.5385+j4.833)(s+1.5385-j4.833)} \right)$$

notes:

- The numerator was chosen so that the DC gain is 150)
- Answers will vary - it's hard to read this graph to 4 decimal places

Block Diagrams

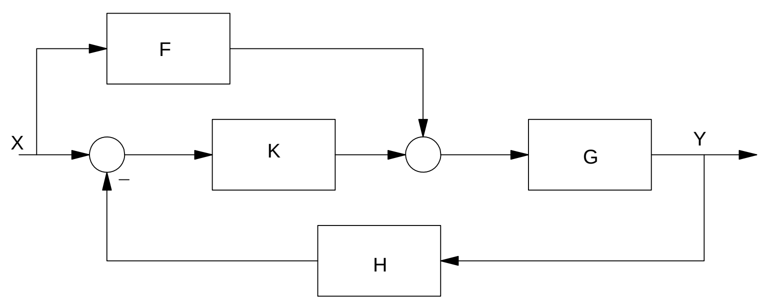
Problem 7) Find the transfer function from X to Y



$$Y = \left(\frac{\sum \text{gains from X to Y}}{1 + \sum \text{Loop Gains}} \right) X$$

$$Y = \left(\frac{AC}{1+A+AB+ACD} \right) X$$

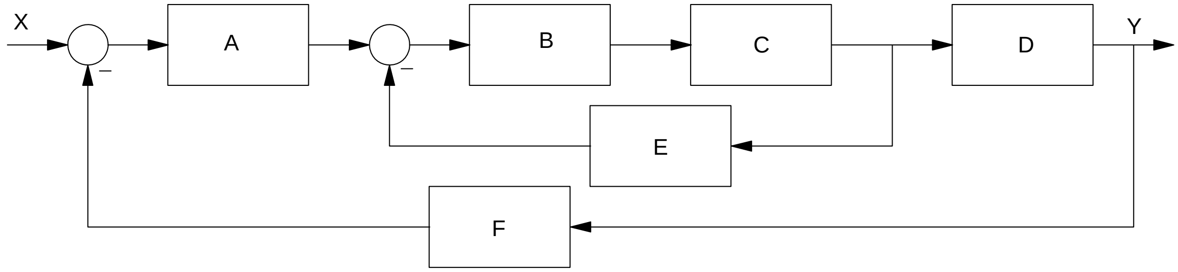
Problem 8: Find the transfer funciton from X to Y



$$Y = \left(\frac{\sum \text{gains from X to Y}}{1 + \sum \text{Loop Gains}} \right) X$$

$$Y = \left(\frac{GF+GK}{1+GKH} \right) X$$

Problem 9: Find the transfer funciton from X to Y



$$Y = \left(\frac{\sum \text{gains from X to Y}}{1 + \sum \text{Loop Gains}} \right) X$$

$$Y = \left(\frac{ABCD}{1 + BCE + ABCDF} \right) X$$