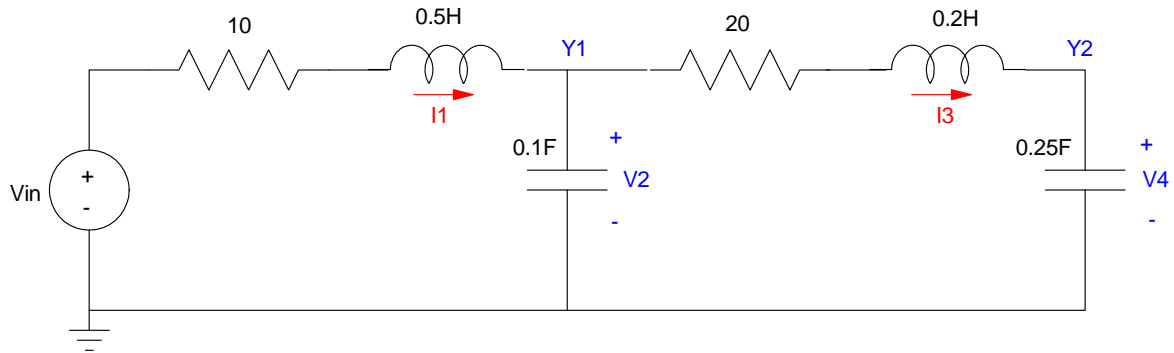


# Solution to Homework #5: ECE 461

State-Space, Canonical Forms, Transfer Functions for Electrical Circuits, Heat Equation

## Transfer Functions for Electrical Circuits

### Problem 1 & 2:



Problem 1 & 2

**Problem 1:** Assume the output is Y1.

Write the N coupled differential equations which describe this circuit (i.e. write the dynamics in terms of IL and Vc)

$$V_1 = 0.5sI_1 = V_{in} - 10I_1 - V_2$$

$$I_2 = 0.1sV_2 = I_1 - I_3$$

$$V_3 = 0.2sI_3 = V_2 - 20I_3 - V_4$$

$$I_4 = 0.25sV_4 = I_3$$

Express these differential equations in state-space form with the state variables being  $\{I_1, V_2, I_3, V_4\}$

Group terms:

$$sI_1 = -20I_1 - 2V_2 + 2V_{in}$$

$$sV_2 = 10I_1 - 10I_3$$

$$sI_3 = 5V_2 - 100I_3 - 5V_4$$

$$sV_4 = 4I_3$$

Place in matrix form

$$s \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} -20 & -2 & 0 & 0 \\ 10 & 0 & -10 & 0 \\ 0 & 5 & -100 & -5 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

Find the transfer function from  $V_{in}$  to Y1

$$Y_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + [0]V_{in}$$

In Matlab:

```
>> A = [-20,-2,0,0 ; 10,0,-10,0 ; 0,5,-100,-5 ; 0,0,4,0 ]
```

```

-20    -2     0     0
 10     0   -10     0
  0     5  -100    -5
  0     0     4     0

```

```
>> B = [2 ; 0 ; 0 ; 0]
```

```

2
0
0
0

```

```
>> C = [0,1,0,0];
```

```
>> D = 0;
```

```
>> G = ss(A,B,C,D);
```

```
>> zpk(G)
```

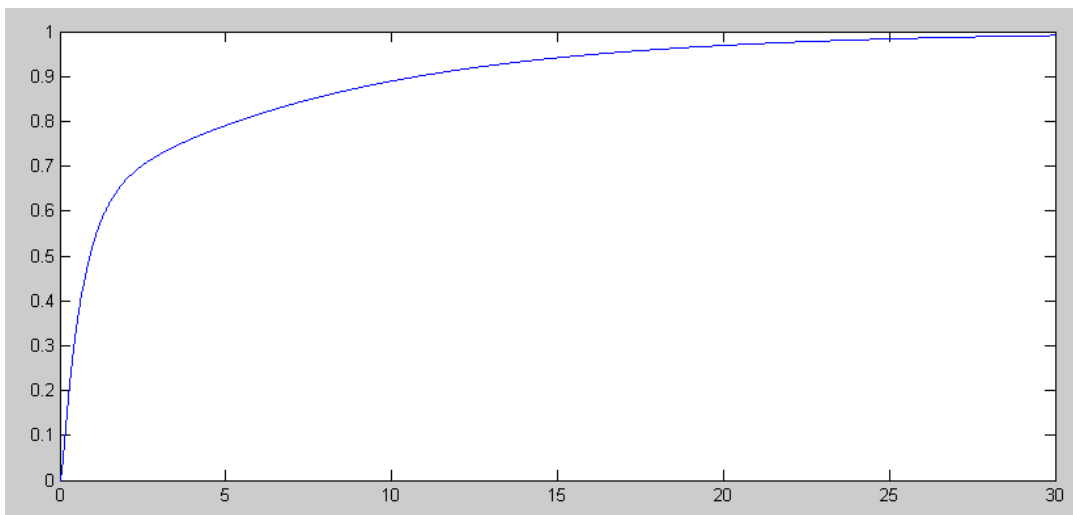
$$G(s) = \frac{20 (s+99.8) (s+0.2004)}{(s+99.3) (s+18.91) (s+1.67) (s+0.1276)}$$

Plot the step response

```
t = [0:0.01:30]';
```

```
y = step(G,t);
```

```
plot(t,y)
```



**Problem 2:** Assume the output is Y2.

Find the transfer function from  $V_{in}$  to Y2

The only thing that changes is the output equation

$$Y_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix} + [0]V_{in}$$

`C = [0,0,0,1];`

`D = 0;`

`G = ss(A,B,C,D);`

`zpk(G)`

$$G(s) = \frac{400}{(s+99.3)(s+18.91)(s+1.67)(s+0.1276)}$$

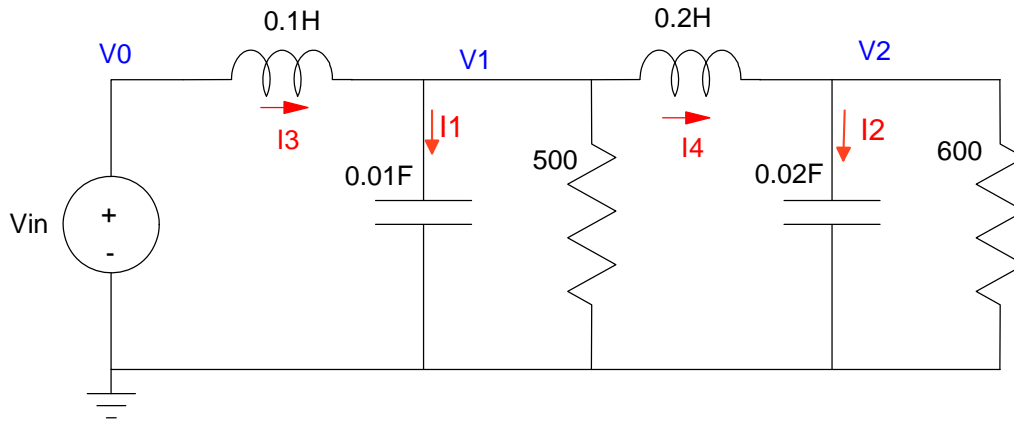
What changes in the state-space equations when you change the output?

- (C, D) change.
- (A, B) determine how the energy flows in the system. Changing the output doesn't change the energy flow.

What changes in the transfer function when you change the output? (do the poles change? do the zeros change?)

- The numerator changes
- The denominator remains unchanged when you change the output.

**Problem 3:**



Write the N coupled differential equations which describe this circuit (i.e. write the dynamics in terms of IL and Vc)

$$I_1 = 0.01sV_1 = I_3 - \left(\frac{V_1}{500}\right) - I_4$$

$$I_2 = 0.02sV_2 = I_4 - \left(\frac{V_2}{600}\right)$$

$$V_3 = 0.1sI_3 = V_{in} - V_1$$

$$V_4 = 0.2sI_4 = V_1 - V_2$$

Group terms

$$sV_1 = 100I_3 - 0.2V_1 - 100I_4$$

$$sV_2 = 50I_4 - 0.0833V_2$$

$$sI_3 = 10V_{in} - 10V_1$$

$$sI_4 = 5V_1 - 5V_2$$

Place in matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -0.2 & 0 & 100 & -100 \\ 0 & -0.0833 & 0 & 50 \\ -10 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \\ 0 \end{bmatrix} V_{in}$$

$$Y = V_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix}$$

Solve for the transfer function in Matlab:

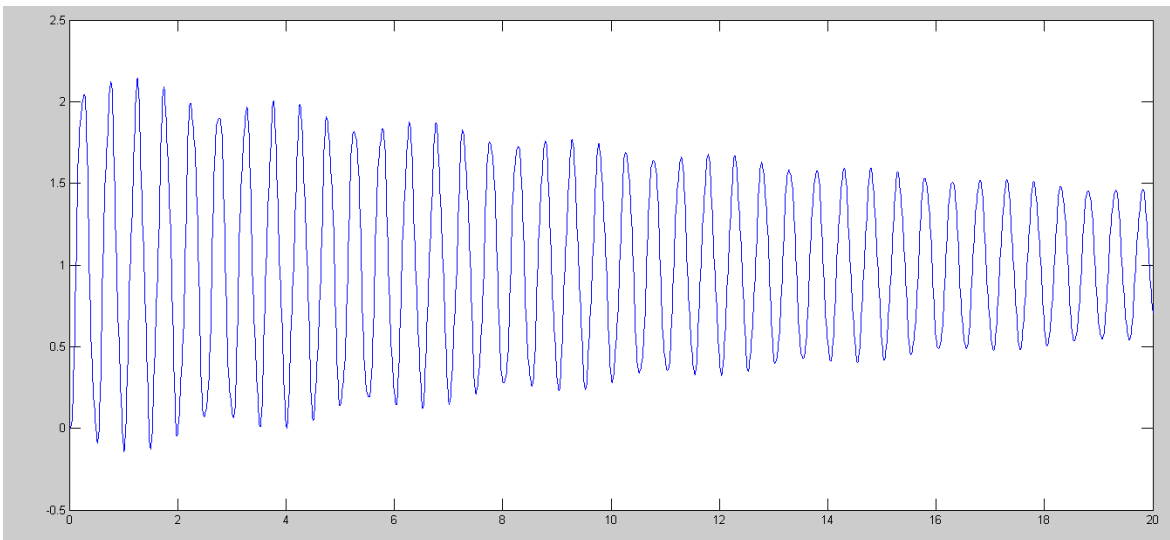
```
A = [-0.2,0,100,-100 ; 0,-0.0833,0,50 ; -10,0,0,0 ; 5,-5,0,0];  
B = [0;0;10;0];  
C = [0,1,0,0];  
D = 0;  
G = ss(A,B,C,D);  
zpk(G)
```

250000

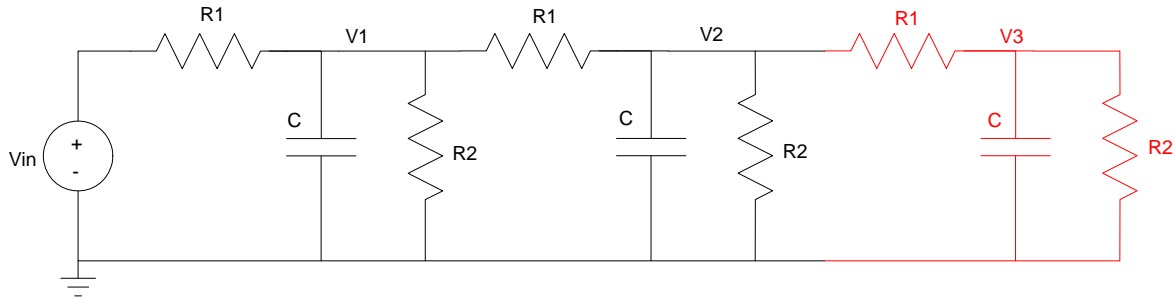
-----  
 $(s^2 + 0.09086s + 156.9) (s^2 + 0.1924s + 1593)$

Plot the step response

```
t = [0:0.01:20]';  
y = step(G,t);  
plot(t,y);
```



**Problem 4: 3-Stage RC Filter**



Problem 4:  $C = 0.01F$ ,  $R_1 = 10 \text{ Ohms}$ ,  $R_2 = 100 \text{ Ohms}$

Write the  $N$  coupled differential equations which describe this circuit (i.e. write the dynamics in terms of  $V_1, V_2, V_3$ )

$$CsV_1 = \left( \frac{V_{in}-V_1}{R_1} \right) + \left( \frac{V_2-V_1}{R_1} \right) + \left( \frac{0-V_1}{R_2} \right)$$

$$CsV_2 = \left( \frac{V_1-V_2}{R_1} \right) + \left( \frac{V_3-V_2}{R_1} \right) + \left( \frac{0-V_2}{R_2} \right)$$

$$CsV_3 = \left( \frac{V_2-V_3}{R_1} \right) + \left( \frac{0-V_3}{R_2} \right)$$

Express these differential equations in state-space form with the state variables being  $\{V_1, V_2, V_3\}$

Group terms

$$sV_1 = \left( \frac{-2}{R_1C} + \frac{-1}{R_2C} \right) V_1 + \left( \frac{1}{R_1C} \right) V_2 + \left( \frac{1}{R_1C} \right) V_{in}$$

$$sV_2 = \left( \frac{-2}{R_1C} + \frac{-1}{R_2C} \right) V_2 + \left( \frac{1}{R_1C} \right) V_1 + \left( \frac{1}{R_1C} \right) V_3$$

$$sV_3 = \left( \frac{-1}{R_1C} + \frac{-1}{R_2C} \right) V_3 + \left( \frac{1}{R_1C} \right) V_2$$

Plug in numbers

$$sV_1 = -21V_1 + 10V_2 + 10V_{in}$$

$$sV_2 = -21V_2 + 10V_1 + 10V_3$$

$$sV_3 = -11V_3 + 10V_2$$

Place in matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -21 & 10 & 0 \\ 10 & -21 & 10 \\ 0 & 10 & -11 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

Find the transfer function from  $V_{in}$  to  $V_3$

$$Y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + [0]V_{in}$$

$$A = [-21, 10, 0 ; 10, -21, 10 ; 0, 10, -11]$$

$$\begin{bmatrix} -21 & 10 & 0 \\ 10 & -21 & 10 \\ 0 & 10 & -11 \end{bmatrix}$$

$$B = [10; 0; 0]$$

$$\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0, 0, 1];$$

$$D = 0;$$

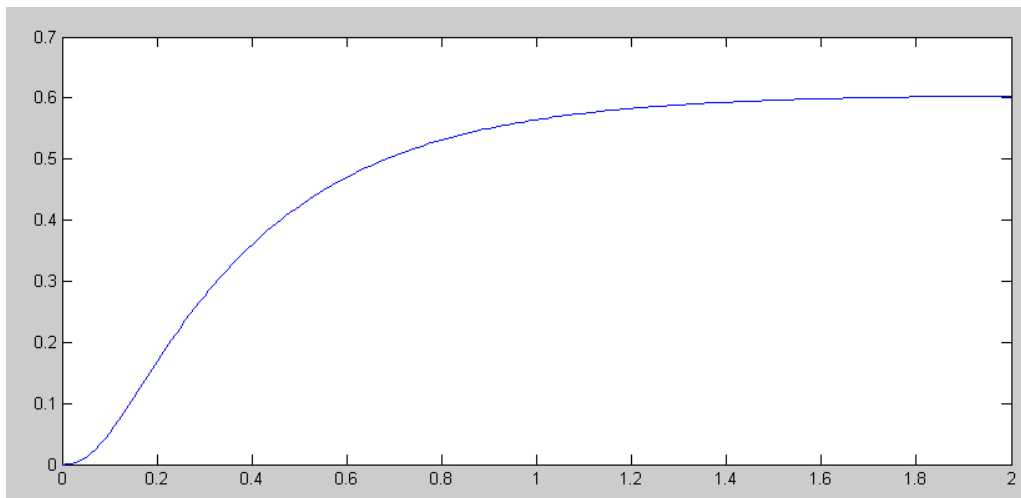
$$G = \text{ss}(A, B, C, D);$$

$$\text{zpk}(G)$$

$$G(s) = \frac{1000}{(s+33.47)(s+16.55)(s+2.981)}$$

Plot the step response

```
t = [0:0.01:2]';  
y = step(G,t);  
plot(t,y);
```



Step Response of 3-Stage RC Filter

**Problem 5:** 10-Stage RC Filter. Add seven more stages to the previous problem (copy the section in red seven more times).

Express these differential equations in state-space form with the state variables being {V1, V2, V3}

```
A = zeros(10,10);
for i=1:9
    A(i,i) = -21;
    A(i,i+1) = 10;
    A(i+1,i) = 10;
end
A(10,10) = -11
```

```
-21    10    0    0    0    0    0    0    0    0
 10   -21   10    0    0    0    0    0    0    0
 0    10   -21   10    0    0    0    0    0    0
 0    0    10  -21   10    0    0    0    0    0
 0    0    0    10  -21   10    0    0    0    0
 0    0    0    0    10  -21   10    0    0    0
 0    0    0    0    0    10  -21   10    0    0
 0    0    0    0    0    0    10  -21   10    0
 0    0    0    0    0    0    0    10  -21   10
 0    0    0    0    0    0    0    0    10  -11
```

```
>> B = zeros(10,1);
>> B(1) = 10
```

```
10
 0
 0
 0
 0
 0
 0
 0
 0
 0
 0
```

```
C = zeros(1,10);
C(10) = 1
```

```
0    0    0    0    0    0    0    0    0    1
```

```
D = 0;
```

Find the transfer function from Vin to V3

```
G = ss(A,B,C,D);
zpk(G)
```

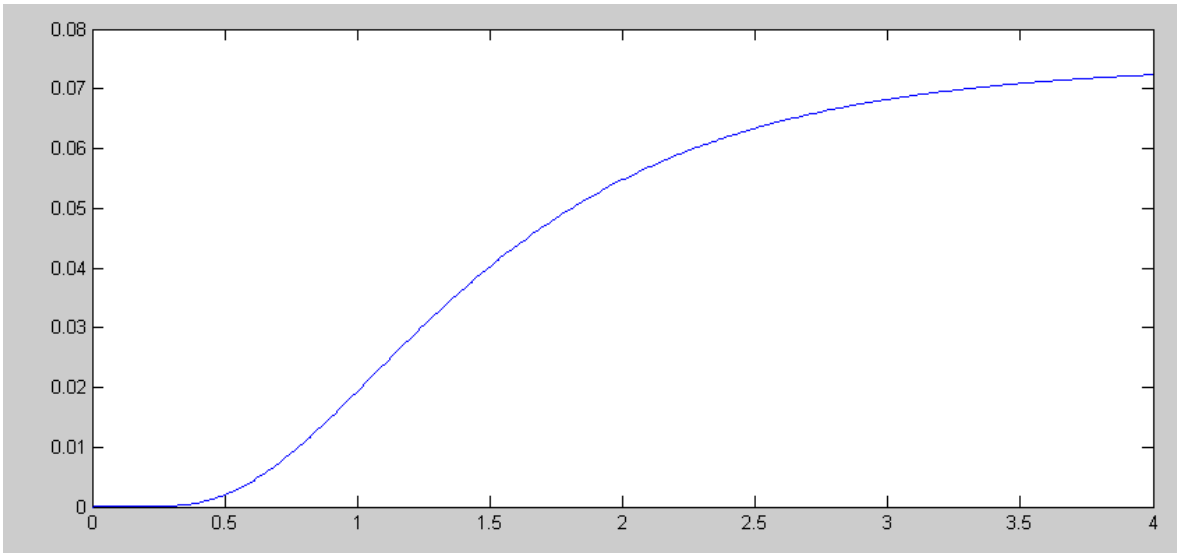
```
G(s) = 
$$\frac{10000000000}{(s+40.11)(s+37.52)(s+33.47)(s+28.31)(s+22.49)(s+16.55)(s+11)(s+6.339)(s+2.981)(s+1.223)}$$

```

Plot the step response

```
>> t = [0:0.01:4]';
>> y = step(G,t);
>> plot(t,y);
```





Step Response of 10-Stage RC Filter