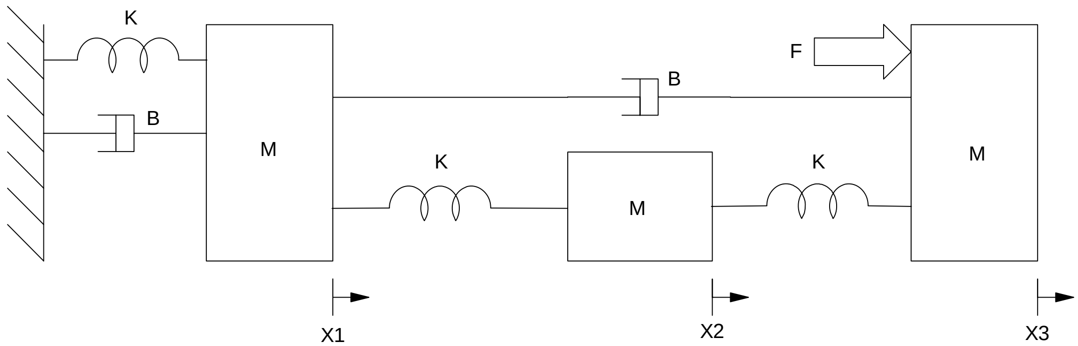
Soltion to Homework #6: ECE 461/661

Mass-Spring Systems, Rotational Systems, DC Servo Motors

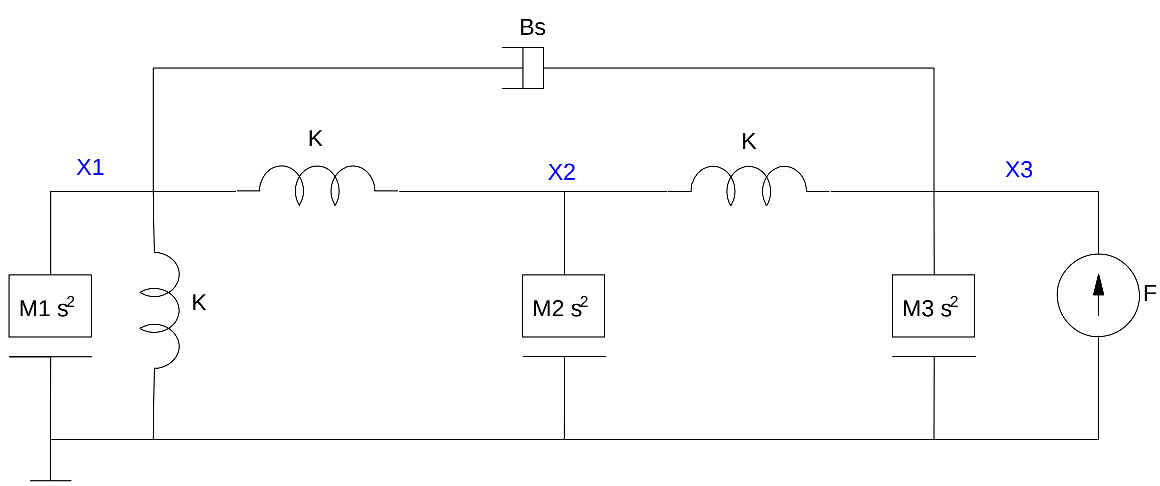
Mass Spring Systems

Problem 1: Assume the output is X3.



Problem 1: M = 1kg, K = 100 N/m, B = 2Ns/m

Draw the circuit equivalent for this mass-spring system



Write the N coupled differential equations which describe this system (i.e. write the votlage node equations)

$$(M_1 s^2 + Bs + 2K)X_1 - (K)X_2 - (Bs)X_3 = 0$$

$$(M_2 s^2 + 2K)X_2 - (K)X_1 - (K)X_3 = 0$$

$$(M_3 s^2 + Bs + K)X_3 - (Bs)X_1 - (K)X_2 = F$$

Place these equations in state-space form. First, solve for the highest derivative

$$s^2 X_1 = \left(\frac{-Bs-2K}{M_1}\right)X_1 + \left(\frac{K}{M_1}\right)X_2 + \left(\frac{Bs}{M_1}\right)X_3$$

$$s^2 X_2 = \left(\frac{-2K}{M_2}\right)X_2 + \left(\frac{K}{M_2}\right)X_1 + \left(\frac{K}{M_2}\right)X_3$$

$$s^2 X_3 = \left(\frac{-Bs-K}{M_3}\right)X_3 + \left(\frac{Bs}{M_3}\right)X_1 + \left(\frac{K}{M_3}\right)X_2 + \left(\frac{1}{M_3}\right)F$$

Plugging in numbers:

$$s^2 X_1 = -200X_1 - 2sX_1 + 100X_2 + 2sX_3$$

$$s^2 X_2 = -200X_2 + 100X_1 + 100X_3$$

$$s^2 X_3 = -100X_3 - 2sX_3 + 2sX_1 + 100X_2 + F$$

Putting in matrix form

$$\begin{bmatrix} sX_1 \\ sX_2 \\ sX_3 \\ s^2X_1 \\ s^2X_2 \\ s^2X_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -200 & 100 & 0 & -2 & 0 & 2 \\ 100 & -200 & 100 & 0 & 0 & 0 \\ 0 & 100 & -100 & 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F$$

$$Y = X_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ sX_1 \\ sX_2 \\ sX_3 \end{bmatrix}$$

Find the transfer function from F to X3

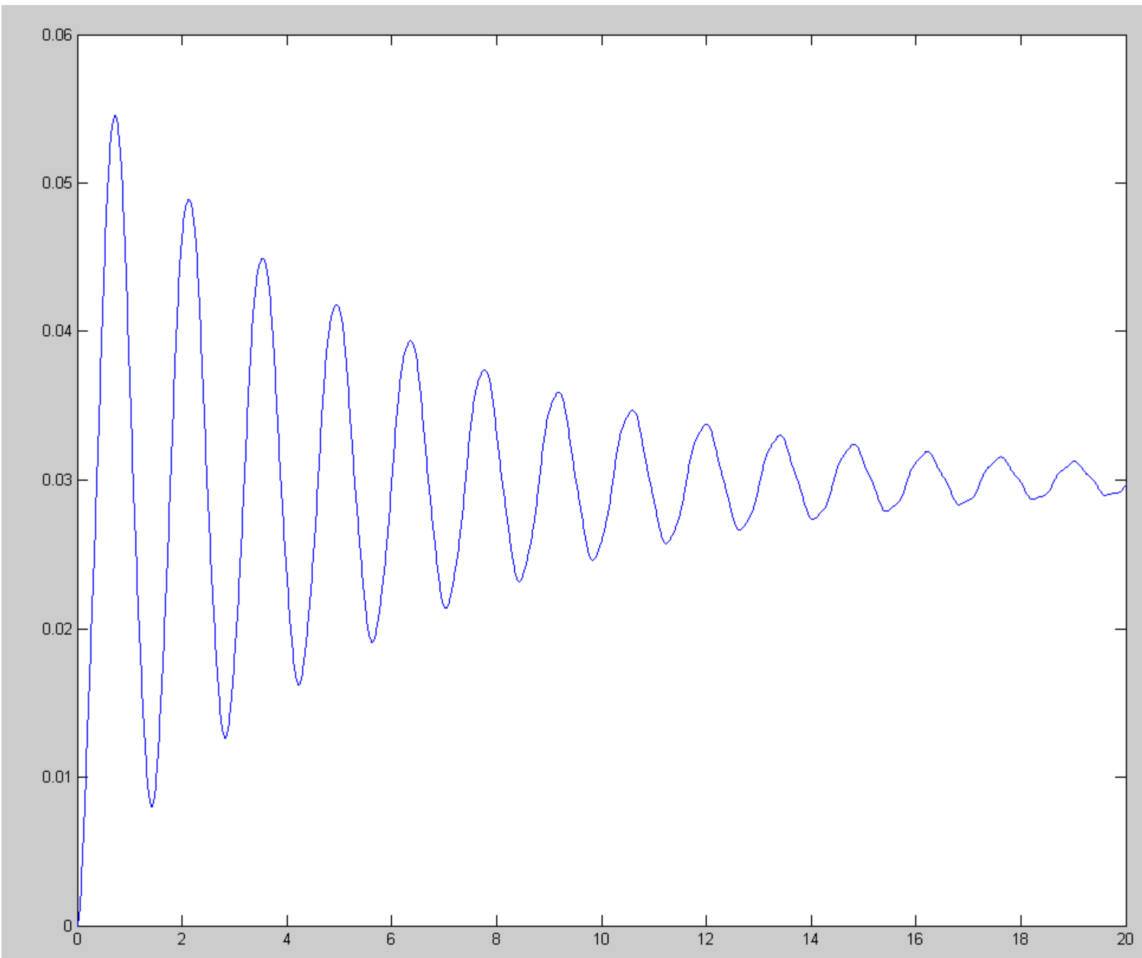
Place in matlab

```
>> A = [0,0,0,1,0,0;0,0,0,0,1,0;0,0,0,0,0,1];
>> A = [A;-200,100,0,-2,0,2;100,-200,100,0,0,0;0,100,-100,2,0,-2]

     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
  -200   100     0    -2     0     2
   100  -200   100     0     0     0
     0   100  -100     2     0    -2
```

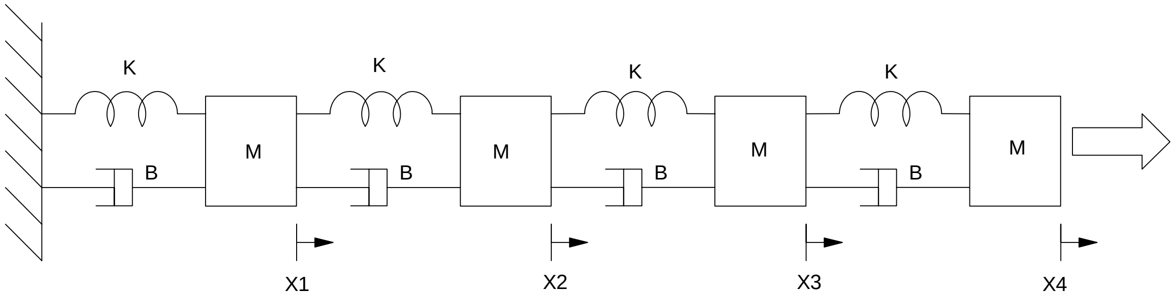
```
>> B = [0;0;0;0;0;1];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

            (s^2 + 1.005s + 100.5) (s^2 + 0.9949s + 298.5)
----------------------------------------------------------------------
(s^2 + 0.3332s + 19.98) (s^2 + 3.547s + 154.5) (s^2 + 0.1197s + 323.8)
```

Plot the step response from F to X3.

```
>> t = [0:0.01:20]';
>> y = step(G,t);
>> plot(t,y);
```

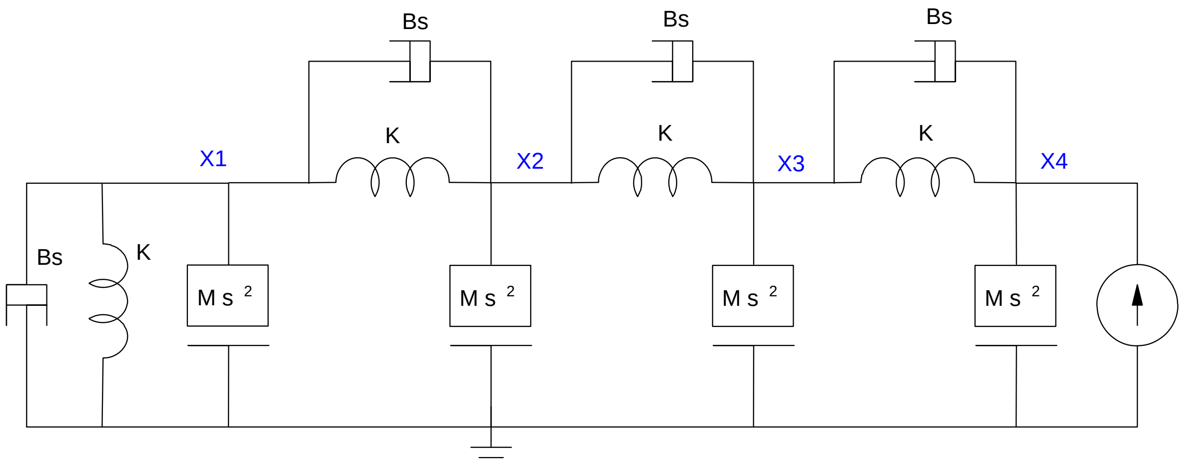


Problem 2: Assume the output is X4.



Problem 2: M = 1kg, K = 100 N/m, B = 2Ns/m

Draw the circuit equivalent for this mass-spring system



Write the N coupled differential equations which describe this system (i.e. write the votlage node equations)

$$(Ms^2 + 2Bs + 2K)X_1 - (Bs + K)X_2 = 0$$

$$(Ms^2 + 2Bs + 2K)X_2 - (Bs + K)X_1 - (Bs + K)X_3 = 0$$

$$(Ms^2 + 2Bs + 2K)X_3 - (Bs + K)X_2 - (Bs + K)X_4 = 0$$

$$(Ms^2 + Bs + K)X_4 - (Bs + K)X_3 = F$$

Find the transfer function from F to X4

Plug in numbers, solve for the highest derivative:

$$s^2 X_1 = (-200 - 4s)X_1 + (100 + 2s)X_2$$

$$s^2 X_2 = (-200 - 4s)X_2 + (100 + 2s)X_1 + (100 + 2s)X_3$$

$$s^2 X_3 = (-200 - 4s)X_3 + (100 + 2s)X_2 + (100 + 2s)X_4$$

$$s^2 X_4 = (-100 - 2s)X_2 + (100 + 2s)X_3 + F$$

Place these equations in state-space form

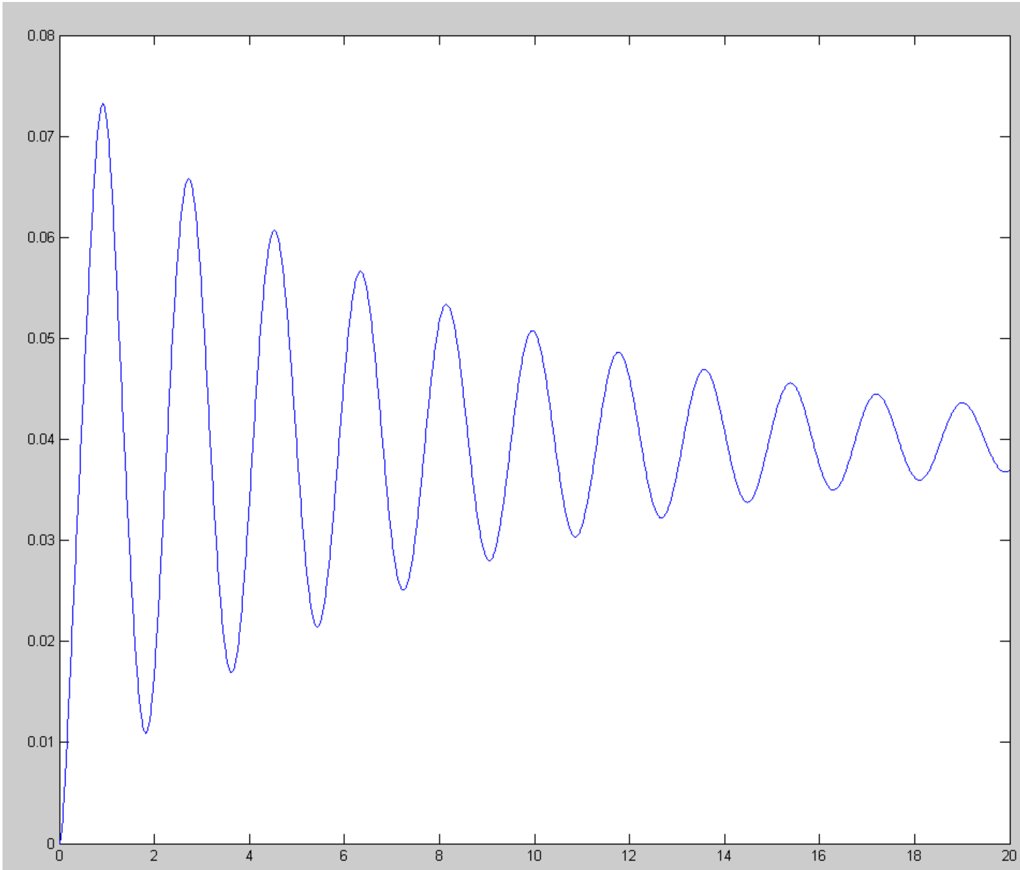
$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & & \\ -200 & 100 & 0 & 0 & -4 & 2 & 0 & 0 \\ 100 & -200 & 100 & 0 & 2 & -4 & 2 & 0 \\ 0 & 100 & -200 & 100 & 0 & 2 & -4 & 2 \\ 0 & 0 & 100 & -100 & 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \cdots \\ sX_1 \\ sX_2 \\ sX_3 \\ sX_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F$$

Place in Matlab

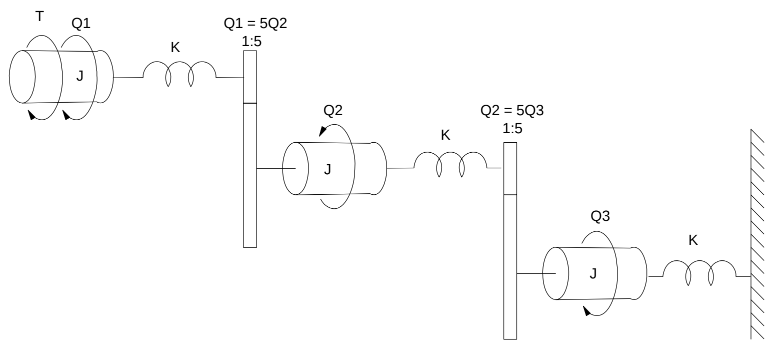
```
>> a11 = zeros(4,4);
>> a12 = eye(4,4);
>> a21 = [-200,100,0,0;100,-200,100,0;0,100,-200,100;0,0,100,-100];
>> a22 = [-4,2,0,0;2,-4,2,0;0,2,-4,2;0,0,2,-2];
>> A = [a11,a12;a21,a22]

A =

     0     0     0     0     1     0     0     0
     0     0     0     0     0     1     0     0
     0     0     0     0     0     0     1     0
     0     0     0     0     0     0     0     1
  -200   100     0     0    -4     2     0     0
   100  -200   100     0     2    -4     2     0
     0   100  -200   100     0     2    -4     2
     0     0   100  -100     0     0     2    -2

>> B = [0;0;0;0;0;0;0;1];
>> C = [0,0,0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)



>> zpk(G)


          (s^2 + 1.172s + 58.58) (s^2 + 4s + 200) (s^2 + 6.828s + 341.4)
---------------------------------------------------------------------------------
(s^2 + 0.2412s + 12.06) (s^2 + 2s + 100) (s^2 + 4.695s + 234.7) (s^2 + 7.064s + 353.2)

>>
```

Plot the step response from F to X4.

```
>> t = [0:0.01:20]';
>> y = step(G,t);
>> plot(t,y);
```

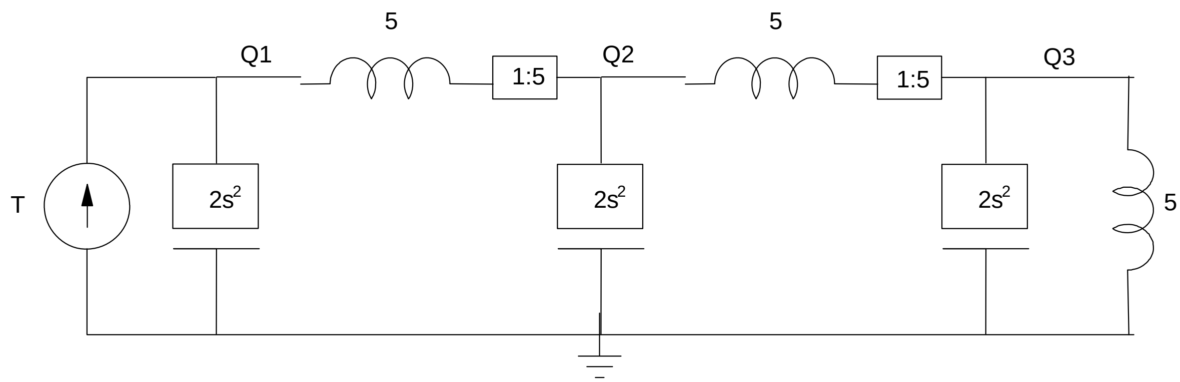

0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
2
4
6
8
10
12
14
16
18
20

Rotational Systems



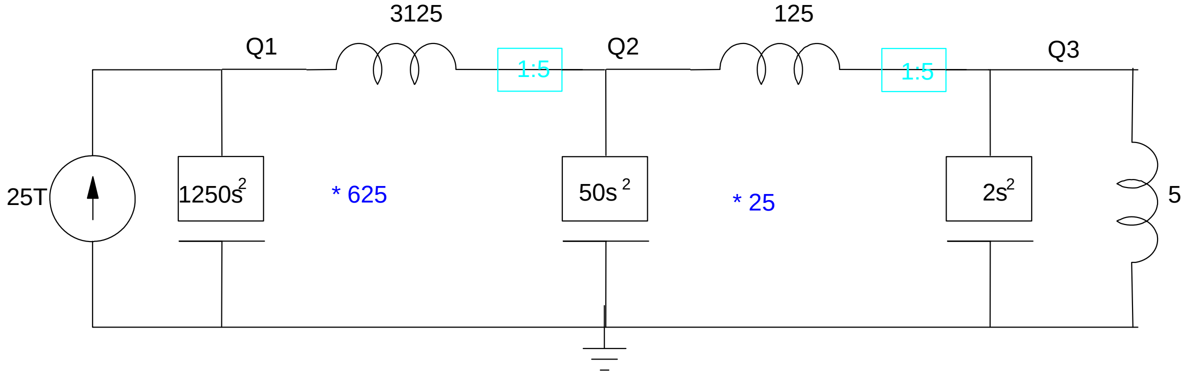
Problem 3: J = 2 Kg m^2, K = 5 Nm/rad

3) Draw the circuit equivalent for the following rotational system



Write the dynamics for this system in state-space form

Remove the gears. Bring everything to Q3 side since that's what we want the output to be



Impedances go through gears as the turn ratio squared

Torque goes through gears as the turn ratio

Now write the voltage node equations

$$(1250s^2+3125)\theta_1-(3125)\theta_2=25T$$

$$(50s^2+3125+125)\theta_2-3125\theta_1+125\theta_3=0$$

$$(2s^2+125+5)\theta_3-125\theta_2=0$$

Solve for the highest derivative

$$s^2\theta_1=-2.5\theta_1+2.5\theta_2+0.02T$$

$$s^2\theta_2=-65\theta_2+62.5\theta_1+2.5\theta_3$$

$$s^2\theta_3=-65\theta_3+62.5\theta_2$$

Place in matrix form

$$\begin{bmatrix} s\theta_1 \\ s\theta_2 \\ s\theta_3 \\ \\ s^2\theta_1 \\ s^2\theta_2 \\ s^2\theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \\ -2.5 & 2.5 & 0 & 0 & 0 & 0 \\ 62.5 & -65 & 2.5 & 0 & 0 & 0 \\ 0 & 62.5 & -65 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \\ s\theta_1 \\ s\theta_2 \\ s\theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0.02 \\ 0 \\ 0 \end{bmatrix} T$$

Place in Matlab

```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-2.5,2.5,0;62.5,-65,2.5;0,62.5,-65];
>> a22 = zeros(3,3);
>> A = [a11,a12;a21,a22]

A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
```

```
   -2.5000    2.5000         0         0         0         0
   62.5000  -65.0000    2.5000         0         0         0
         0   62.5000  -65.0000         0         0         0

>> B = [0;0;0;0.02;0;0];
>> C = [0,0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                   78.125
--------------------------------------------
(s^2 + 0.003688) (s^2 + 53.93) (s^2 + 78.57)
```

DC Servo Motors

4) Determine the transfer function and step response for the following DC servo motor:

Baldour MT-3363-B DC Servo Motor: (476W)

- Rotor Inertia: 3.67 kg cm2
- Viscous Damping: 7.8E-3 Nm/krpm
- Torque Constant: 0.297 Nm/A
- Resistance: 2.4 Ohms
- Inductance: 6.1mH
- Total Weight: 5kg (11 lb)
- Price: $625 on ebay

ebay listing: Baldor MTB-3363-BLYCN servo motor servomotor w/brake
Date Sheets: http://www.baldor.com/mvc/DownloadCenter/Files/BR1202-F

Baldour MT-3363-B DC Servo Motor: (476W)

$$J = 3.67 \text{ kg cm}^2 \cdot \left(\frac{1m}{100cm}\right)^2 = 367 \cdot 10^6 \text{ kg m}^2$$

$$D = 7.8e - 3Nm/krpm \cdot \left(\frac{krev}{1000rev}\right)\left(\frac{1rev}{2\pi rad}\right)\left(\frac{60s}{\min}\right) = 74.48 \cdot 10^{-6} \ \frac{Nm}{rad/\sec}$$

$$K_t = 0.297\frac{Nm}{A}$$

- Resistance: 2.4 Ohms
- Inductance: 6.1mH
- Total Weight: 5kg (11 lb)
- Price: $625 on ebay

```
>> J = 367e-6;
>> D = 74.48e-6;
>> Kt = 0.297;
>> R = 2.4;
>> L = 6.1e-3;
```

The transfer function for a DC motor is

$$s\theta = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right)V_a$$

$$s\theta = \left(\frac{K_t}{(JL)s^2+(JR+DL)s+(DR+K_t^2)}\right)V_a$$

```
>> num = Kt;
>> den = [J*L, J*R + D*L, D*R + Kt^2];
>> G = tf(num,den);
>> zpk(G)
```

132666.2795

(s^2 + 393.6s + 3.948e004)

```
>> t = [0:0.001:1]' * 0.04;
>> y = step(G,t);
>> plot(t,y);
```