ECE 461/661 - Final Exam: Name __Solution__

Fall 2018

1) Assume X and Y are related by the following transfer function

$$Y = \left(\frac{20}{(s+2)(s+5)}\right)X$$

1a) Find y(t) assuming

$$x(t) = 2 + 3\cos(4t)$$

This is an input which has been around for all time. This means

- You're looking for the steady-state solution,
- You should use phasors to solve the problem

(LaPlace transforms also work but is a LOT harder)

Use superposition

x(t) = 2

X = 2 (phasor form)

s = 0

$$\left(\frac{20}{(s+2)(s+5)}\right)_{s=0} = 2$$

$$Y = 2 \cdot 2$$

$$Y = 4$$

which means

$$y(t) = 4$$

x(t) = 3 cos(4t)

X = 3 + j0 (phasor form)

s = j4

$$\left(\frac{20}{(s+2)(s+5)}\right)_{s=j4} = 0.698\angle -102^0$$

$$Y = (0.698\angle -102^0)(3 + j0)$$

$$Y = 2.095\angle -102^0$$

which means

$$y(t) = 2.095\cos(4t - 102^0)$$

The total answer is then

$$y(t) = 4 + 2.095\cos(4t - 102^0)$$

1b) Find y(t) assuming

$$x(t) = 2u(t) = \begin{cases} 2 & t > 0 \\ 0 & otherwise \end{cases}$$

This is a system which turned on at time equals zero, meaning

- The transient response is important
- You need to use LaPlace transforms to solve

Convert to LaPlace

$$Y = \left(\frac{20}{(s+2)(s+5)}\right) X$$

$$Y = \left(\frac{20}{(s+2)(s+5)}\right)\left(\frac{2}{s}\right)$$

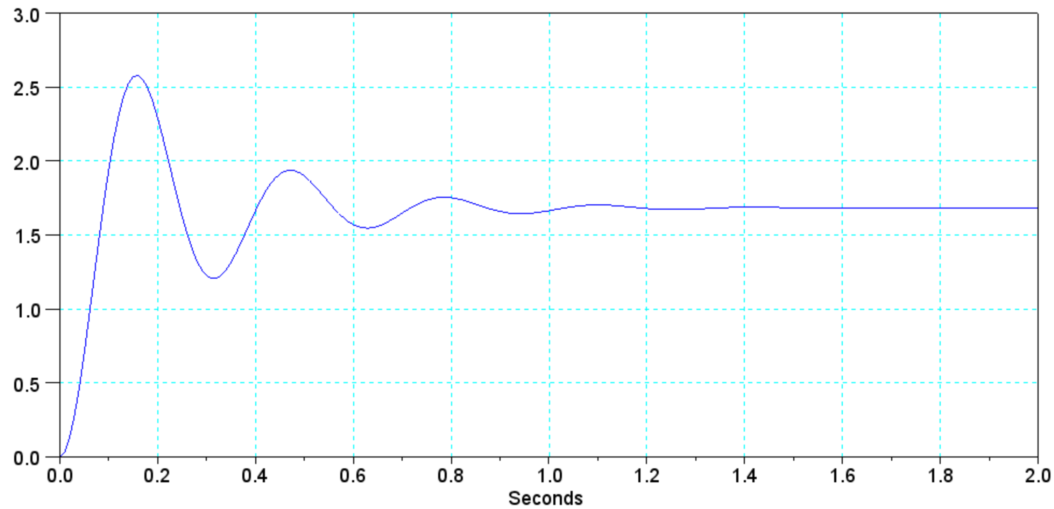
use partial fractions

$$Y = \left(\frac{4}{s}\right) + \left(\frac{-6.667}{s+2}\right) + \left(\frac{2.667}{s+5}\right)$$

Take the inverse LaPlace transform

$$\boxed{y(t) = 4 - 6.667e^{-2t} + 2.667e^{-5t}} \qquad t > 0$$

2a) Determine the transfer function for the system with the following step response



This is a 2nd-order system (it oscillates, meaning energy is bouncing back and forth between two states). To find the transfer funciton, you need three terms

DC Gain: 1.7

Frequency of Oscillation:

$$\omega = 2\pi f = 2\pi\left(\frac{3 \text{ cycles}}{0.92 \text{ seconds}}\right) = 20.49 \quad \text{rad/sec}$$

Settling Time: Ts = 1.1 second (approx)

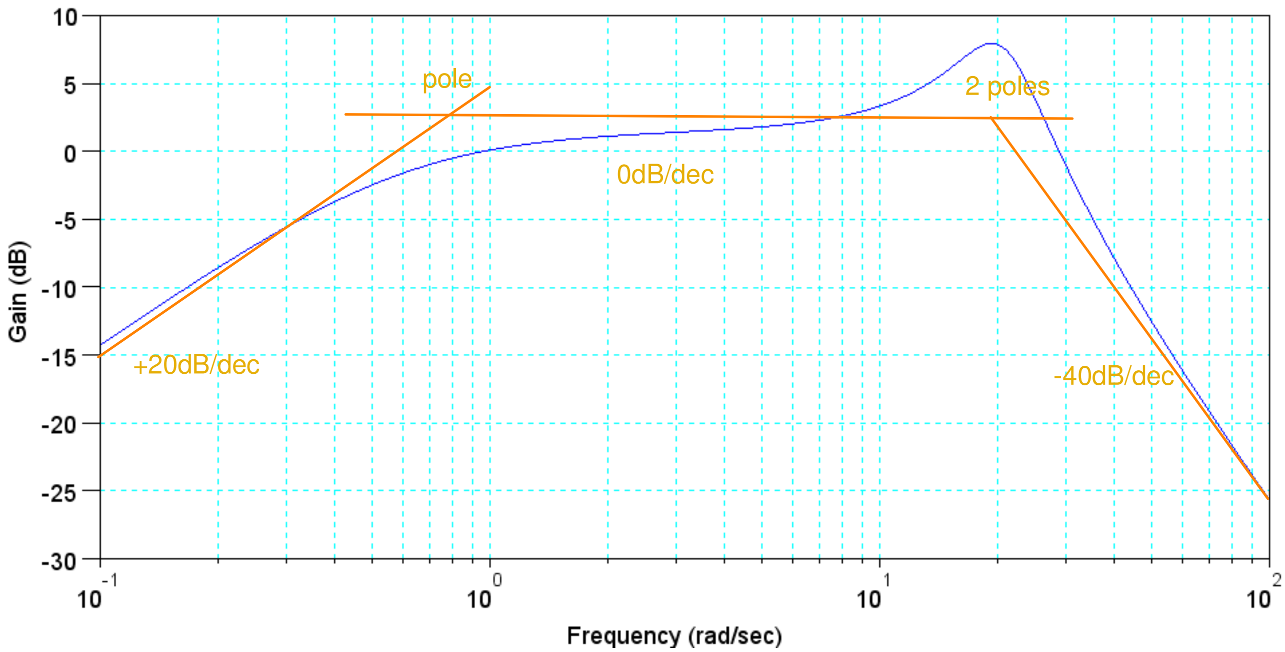
$$\sigma = \frac{4}{T_s} = 3.63$$

So, the transfer funciton is

$$G(s) \approx \left(\frac{736}{(s+3.63+j20.49)(s+3.63-j20.49)}\right)$$

(the numerator is whatever it took to make the DC gain equal to 1.7)

2b) Determine the transfer function for the system with the following frequency response



First, draw in the straight-line approximations. Make sure each slope is a multiple of 20dB/decade

The corners mark the poles and zeros

- There is a zero left of 0.1. Assume the zero is at s = 0
- There is a pole at 0.8 rad/sec
- There are two poles at 20 rad/sec

For the two poles, the angle comes from

gain at corner = +9dB = 2.818

$$\frac{1}{2\zeta} = 2.818$$

$$\zeta = 0.177$$

$$\theta = \arccos(\zeta) = 79.8^0$$

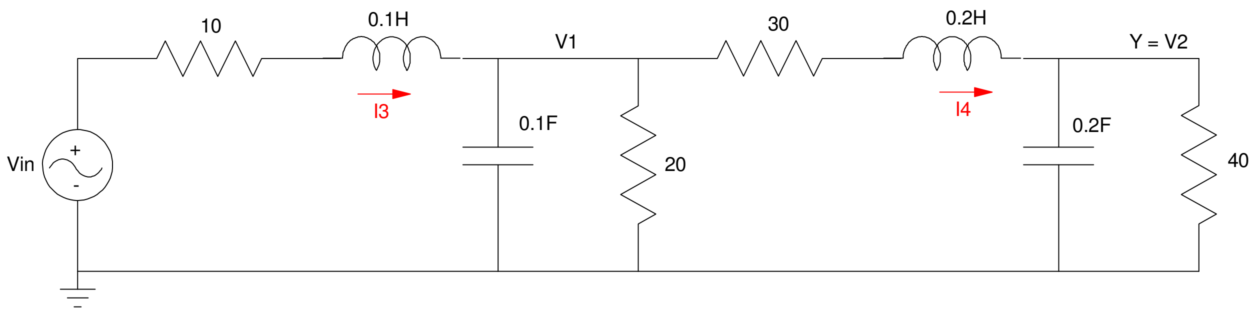
so

$$G(s) \approx \left(\frac{399s}{(s+0.8)\left(s+20\angle \pm 79.8^0\right)} \right)$$

To find the gain (399) pick any point. At s = j1, the gain is 0db = 1

$$\left(\frac{ks}{(s+0.8)\left(s+20\angle \pm 79.8^0\right)} \right)_{s=j1} = 1$$

3a) Write four coupled differential equations to describe the dynamics of this circuit



$$I_1 = 0.1\dot{V}_1 = I_3 - I_4 - \frac{V_1}{20}$$

$$I_2 = 0.2\dot{V}_2 = I_4 - \frac{V_2}{40}$$

$$V_3 = 0.1\dot{I}_3 = V_{in} - 10I_3 - V_1$$

$$V_4 = 0.2\dot{I}_4 = V_1 - 30I_4 - V_2$$

3b) Express these dynamics in state-space form

$$\dot{V}_1 = 10I_3 - 10I_4 - 0.5V_1$$

$$\dot{V}_2 = 5I_4 - 0.125V_2$$

$$\dot{I}_3 = 10V_{in} - 100I_3 - 10V_1$$

$$\dot{I}_4 = 5V_1 - 150I_4 - 5V_2$$

In matrix form

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{I}_4 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 10 & -10 \\ 0 & -0.125 & 0 & 5 \\ -10 & 0 & -100 & 0 \\ 5 & -5 & 0 & -150 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \\ 0 \end{bmatrix} V_{in}$$

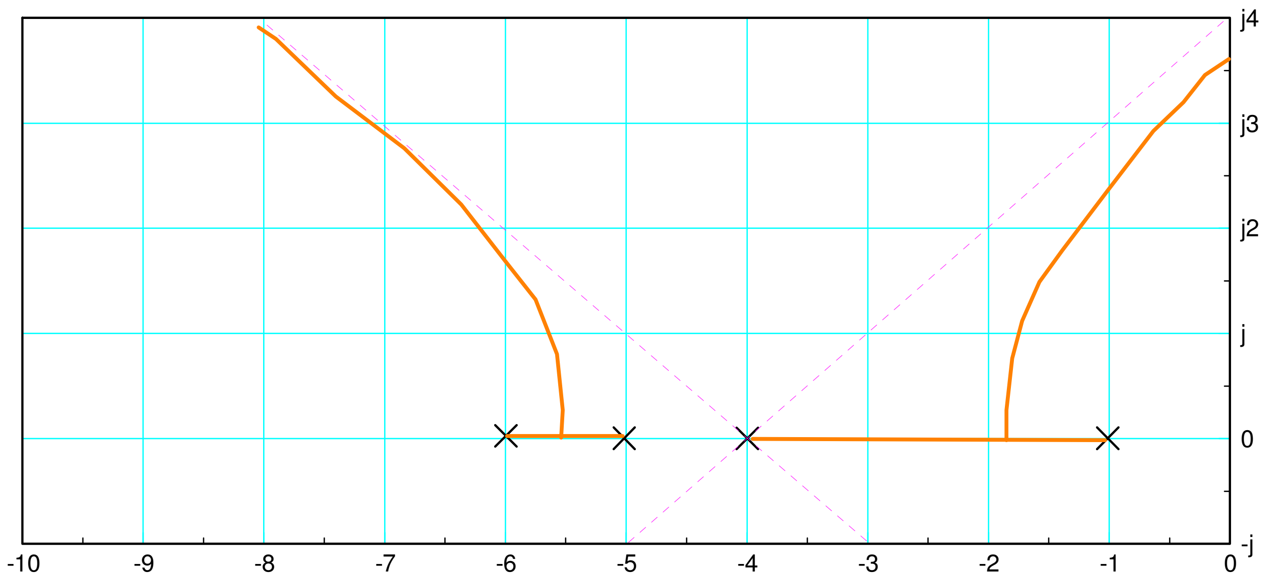
$$Y = V_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} + [0]V_{in}$$

4) Sketch the root locus for

$$G(s) = \left(\frac{100}{(s+1)(s+4)(s+5)(s+6)}\right)$$

Determine the following:

Real Axis Loci	(-1, -4) (-5, -6)
Breakaway Points (approx)	-1.941, -5.605
jw Crossing (approx)	j3.482
Asymptotes	show on graph



There are 4 poles and no zeros, meaning there are 4 asymptotes

$$\text{Angle} = \frac{180^0 + N \cdot 360^0}{4} = \{\pm 45^0, \pm 135^0\}$$

The intersect is the center of mass

$$\left(\frac{-1-4-5-6}{4}\right) = -4$$

5) Assume a system has the following dynamics:

$$G(s) = \left(\frac{100}{(s+2)(s+4)(s+10)}\right)$$

Design a compensator, K(s), so that the closed-loop system has

- No error for a step input, and
- Closed-loop dominant poles at **s = -3 + j2**

(note: this implies $GK(-3+j2) = 1\angle 180^0$)

Pick K(s) to be of the form

$$K = \left(\frac{k(s+2)(s+4)}{s(s+a)}\right)$$

Find 'a' so that -3 + j2 is on the root locus (angles add up to 180 degrees). Taking the part you know:

$$GK = \left(\frac{100}{s(s+10)}\right)_{s=-3+j2} = 3.810\angle -162.2^0$$

To make the angles add up to 180 degrees

$$\angle(s+a) = 17.745^0$$

$$a = \frac{2}{\tan\left(17.745^0\right)} + 3 = 9.25$$

This results in

$$GK = \left(\frac{100}{s(s+9.25)(s+10)}\right)_{s=-3+j2} = 0.581\angle 180^0$$

Pick 'k' to make the gain one

$$k = \frac{1}{0.581} = 17.23$$

and

$$\boxed{K(s) = \left(\frac{17.23(s+2)(s+4)}{s(s+9.25)}\right)}$$

6) Assume a system has the following dynamics:

$$G(s) = \left(\frac{100}{(s+2)(s+4)(s+10)}\right)$$

Design a digital compensator, K(z), so that the closed-loop system has

- No error for a step input,
- Closed-loop dominant poles at **s = -3 + j2**, and

Assume a sampling rate of T = 100ms (T = 0.1)

Solve as

$$G(s) \cdot e^{-sT/2} \cdot K(z) = 1\angle 180^0$$

Let

$$K(s) = \left(\frac{k(s+2)(s+4)}{s(s+a)}\right)$$

Converting to the z-plane (conversion is $z = e^{sT}$)

$$K(z) = \left(\frac{k(z-0.819)(z-0.670)}{(z-1)(z-b)}\right)$$

Evaluate what you know

$$\left(\left(\frac{100}{(s+2)(s+4)(s+10)}\right) \cdot e^{-sT/2}\right)_{s=-3+j2} \cdot \left(\frac{(z-0.819)(z-0.670)}{(z-1)}\right)_{z=0.726+j0.147} = 0.281\angle -162^0$$

For the angles to add up to 180 degrees

$$\angle(z-b) = 17.997^0$$

$$b = 0.726 - \frac{0.147}{\tan\left(17.997^0\right)} = 0.274$$

Evaluating the whole mess again

$$\left(\left(\frac{100}{(s+2)(s+4)(s+10)}\right) \cdot e^{-sT/2}\right)_{s=-3+j2} \cdot \left(\frac{(z-0.819)(z-0.670)}{(z-1)(z-0.274)}\right)_{z=0.726+j0.147} = 0.592\angle 180^0$$

k makes the gain one

$$k = \frac{1}{0.592} = 1.690$$

$$K(z) = \left(\frac{1.690(z-0.819)(z-0.670)}{(z-1)(z-0.274)}\right)$$

7) Assume a system has the following dynamics:

$$G(s) = \left(\frac{100}{(s+2)(s+4)(s+10)}\right)$$

Design a compensator, K(s), so that the closed-loop system has

- A DC gain of one (no error for a step input),
- A 0dB gain frequency of 5 rad/sec, and
- A phase margin of 50 degrees

(note: this implies $GK(j5) = 1\angle -130^0$)

Let

$$K(s) = \left(\frac{k(s+2)(s+4)}{s(s+a)}\right)$$

then

$$GK = \left(\frac{100k}{s(s+10)(s+a)}\right)_{s=j5} = 1\angle -130^0$$

Evaluate what you know

$$\left(\frac{100}{s(s+10)}\right)_{s=j5} = 11.7889\angle -116.5^0$$

For the angles to add up to -130 degrees

$$\angle(s+a) = 13.43^0$$

$$a = \frac{5}{\tan\left(13.43^0\right)} = 20.93$$

So now

$$\left(\frac{100}{s(s+10)(s+20.93)}\right)_{s=j5} = 0.0831\angle -130^0$$

$$k = \frac{1}{0.0831} = 12.029$$

and

$$\boxed{K(s) = \left(\frac{12.029(s+2)(s+4)}{s(s+20.93)}\right)}$$

8a) Determine a discrete-time equivalent to K(s). Assume a sampling rate of 100ms (T = 0.1)

$$K(s) = \left(\frac{20(s+3)(s+5)}{s(s+10)}\right)$$

s = -3

$$z = e^{sT} = e^{-3T} = 0.7408$$

s = -5

$$z = e^{-5T} = 0.6065$$

s = 0

$$z = e^{0T} = 1$$

s = -10

$$z = e^{-10T} = 0.3679$$

so

$$K(z) = \left(\frac{(z-0.7408)(z-0.6065)}{(z-1)(z-0.3679)}\right)$$

Match the gain at some frequency. DC doesn't work (the DC gain is infinity), so pick s = j0.1

$$\left(\frac{20(s+3)(s+5)}{s(s+10)}\right)_{s=j0.1} = 300.21\angle -87.5^0$$

$$\left(\frac{(z-0.7408)(z-0.6065)}{(z-1)(z-0.3679)}\right)_{z=e^{j0.1T}} = 16.147\angle -87.5^0$$

The gain is off by 18.59, so

$$K(z) = \left(\frac{18.59(z-0.7408)(z-0.6065)}{(z-1)(z-0.3679)}\right)$$

8b) Design a circuit to implement K(s)

$$K(s) = \left(\frac{20(s+3)(s+5)}{s(s+10)}\right) = \left(\frac{2(s+3)}{s}\right)\left(\frac{10(s+5)}{s+10}\right)$$