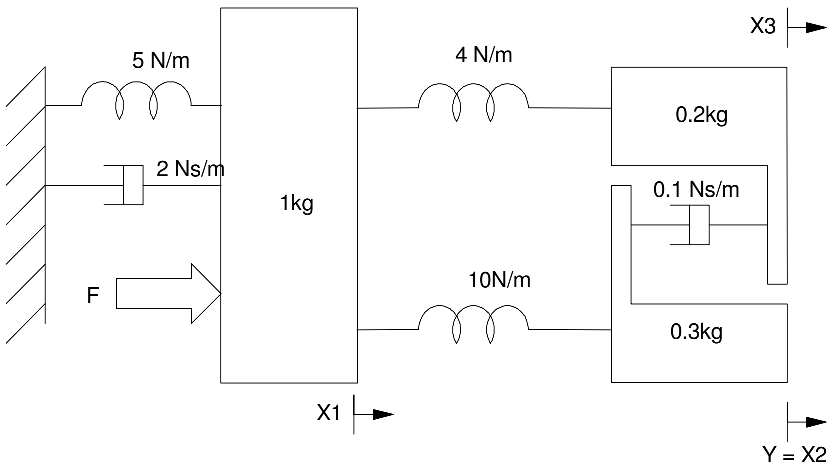
Homework #5: ECE 461/661

Mass-Spring Systems, Rotational Systems.   Due Monday, September 28th

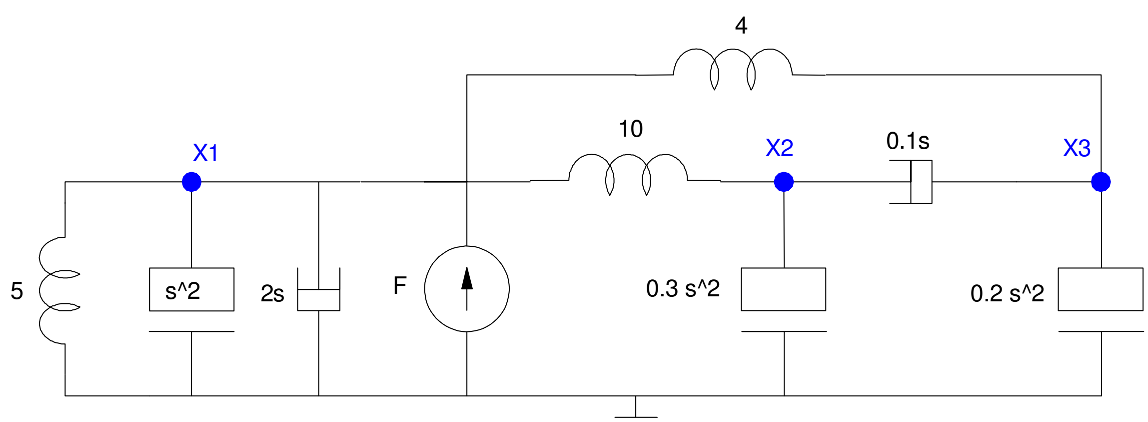
Mass Spring systems

1) Draw the circuit equivalent for the following mass-spring systems.
   - Express the dynamics in state-space form
   - Find the transfer function from F to X2
   - Plot the step response from F to X2



Problem 1

First, draw the circuit equivalent



Write the voltge node equations

$$(s^2 + 2s + 10 + 4 + 5)X_1 - 10X_2 - 4X_3 = F$$

$$(0.3s^2 + 0.1s + 10)X_2 - 10X_1 - 0.1sX_3 = 0$$

$$(0.2s^2 + 0.1s + 4)X_3 - 4X_1 - 0.1sX_2 = 0$$

Group terms and solve for the highest derivative

$$s^2 X_1 = (-2s - 19)X_1 + 10X_2 + 4X_3 + F$$

$$s^2 X_2 = (-0.333s - 33.333)X_2 + 33.333X_1 + 0.333sX_3$$

$$s^2 X_3 = (-0.5s - 20)X_3 + 20X_1 + 0.5sX_2$$

Place in matrix form

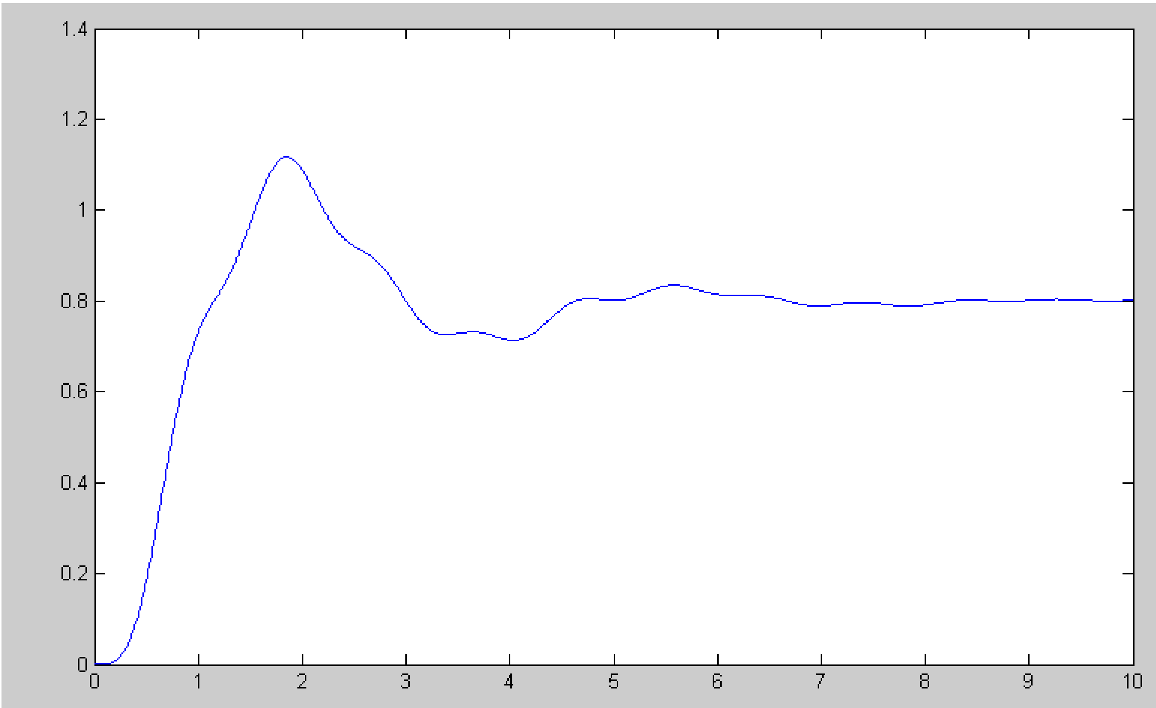
$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -19 & 10 & 4 & -2 & 0 & 0 \\ 33.333 & -33.333 & 0 & 0 & -0.333 & 0.333 \\ 20 & 0 & -20 & 0 & 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ sx_1 \\ sx_2 \\ sx_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}F$$

Find the transfer funciton to X2

```
>> A = [0,0,0,1,0,0;
0,0,0,0,1,0;
0,0,0,0,0,1;
-19,10,4,-2,0,0;
33.333,-33.333,0,0,-0.333,0.333;
20,0,-20,0,0.5,-0.5]

A =

         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
  -19.0000   10.0000    4.0000   -2.0000         0         0
   33.3330  -33.3330         0         0   -0.3330    0.3330
   20.0000         0  -20.0000         0    0.5000   -0.5000

>> B = [0;0;0;4;0;0]

B =

     0
     0
     0
     4
     0
     0

>> C = [0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

                    133.332 (s^2 + 0.6998s + 20)
-------------------------------------------------------------------
(s^2 + 1.245s + 3.267) (s^2 + 0.8336s + 22.23) (s^2 + 0.7544s + 45.9)

>>
```

Plotting the step response

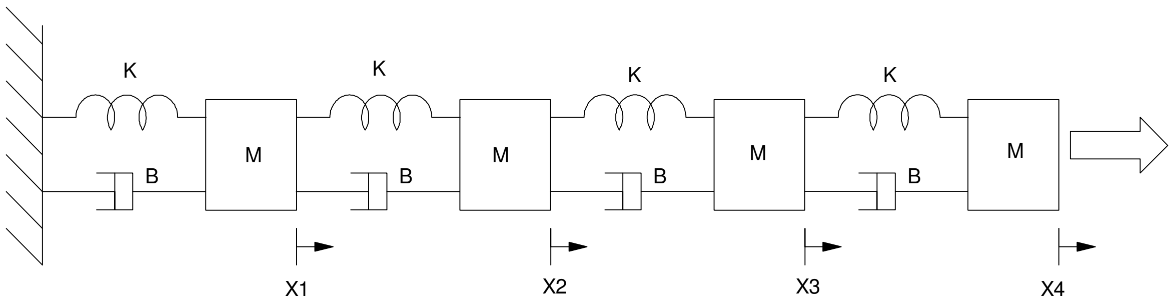
```
>> t = [0:0.01:10]';
>> y = step(G,t);
>> plot(t,y);
```



2) Draw the circuit equivalent for the following mass-spring systems.

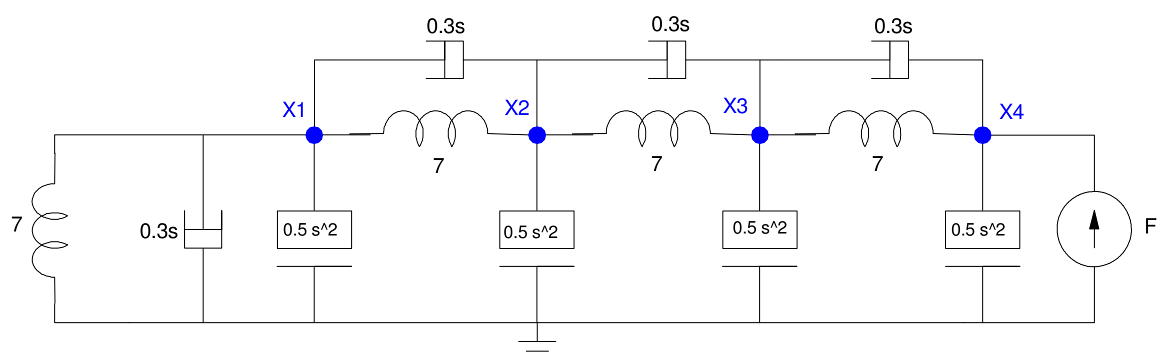
- Express the dynamics in state-space form
- Find the transfer function from F to X4

Plot the step response from F to X4



Problem 6: M = 0.5kg, B = 0.3 Ns/m, K = 7 N/m

First, draw the circuit equivalent



Next, write the voltge node equations

$$(0.5s^2 + 0.6s + 14)X_1 - (0.3s + 7)X_2 = 0$$

$$(0.5s^2 + 0.6s + 14)X_2 - (0.3s + 7)X_1 - (0.3s + 7)X_3 = 0$$

$$(0.5s^2 + 0.6s + 14)X_3 - (0.3s + 7)X_2 - (0.3s + 7)X_4 = 0$$

$$(0.5s^2 + 0.3s + 7)X_4 - (0.3s + 7)X_3 = F$$

Solve for the highest derivative

$$s^2X_1 = (-1.2s - 28)X_1 + (0.6s + 14)X_2$$

$$s^2X_2 = (-1.2s - 28)X_2 + (0.6s + 14)X_1 + (0.6s + 14)X_3$$

$$s^2X_3 = (-1.2s - 28)X_3 + (0.6s + 14)X_2 + (0.6s + 14)X_4$$

$$s^2X_4 = (-0.6s - 14)X_4 + (0.6s + 14)X_3 + 2F$$

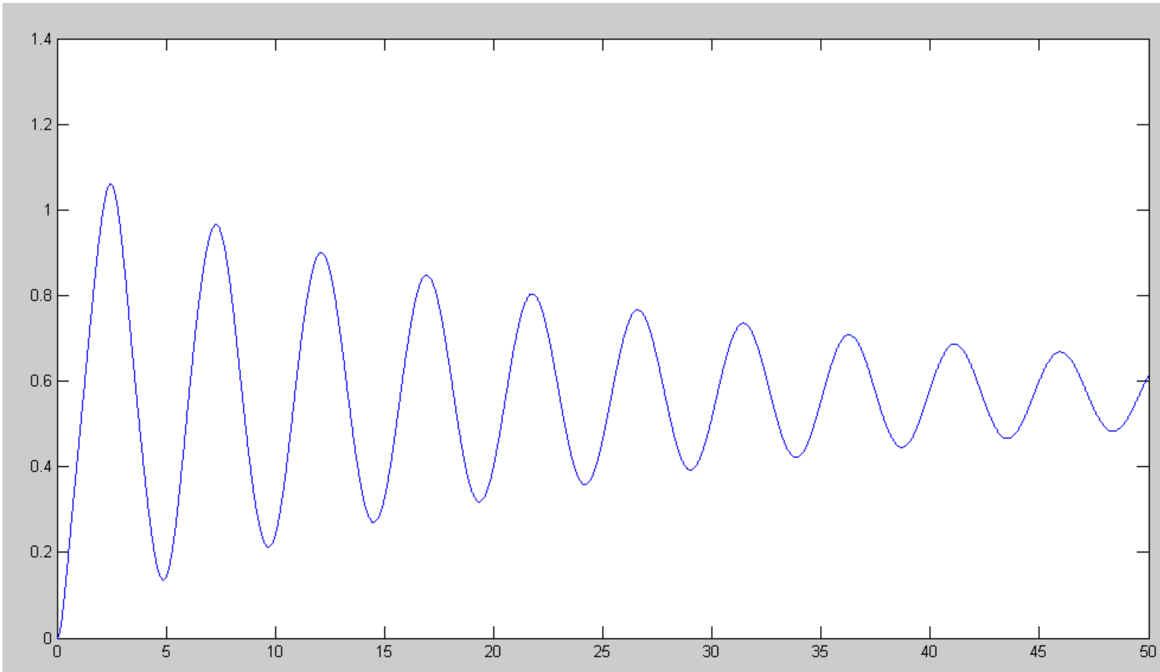
Place in matrix form

$$s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ sx_1 \\ sx_2 \\ sx_3 \\ sx_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -28 & 14 & 0 & 0 & -1.2 & 0.6 & 0 & 0 \\ 14 & -28 & 14 & 0 & 0.6 & -1.2 & 0.6 & 0 \\ 0 & 14 & -28 & 14 & 0 & 0.6 & -1.2 & 0.6 \\ 0 & 0 & 14 & -14 & 0 & 0 & 0.6 & -0.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ sx_1 \\ sx_2 \\ sx_3 \\ sx_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}F$$

Place in Matlab

```
>> a11 = zeros(4,4);
>> a12 = eye(4,4);
>> a21 = [-28,14,0,0;14,-28,14,0;0,14,-28,14;0,0,14,-14]
>> a22 = [-1.2,0.6,0,0;0.6,-1.2,0.6,0;0,0.6,-1.2,0.6;0,0,0.6,-0.6];
>> A = [a11,a12 ; a21, a22]

          0          0          0          0     1.0000          0          0          0
          0          0          0          0          0     1.0000          0          0
          0          0          0          0          0          0     1.0000          0
          0          0          0          0          0          0          0     1.0000
   -28.0000    14.0000          0          0    -1.2000     0.6000          0          0
    14.0000   -28.0000    14.0000          0     0.6000    -1.2000     0.6000          0
          0    14.0000   -28.0000    14.0000          0     0.6000    -1.2000     0.6000
          0          0    14.0000   -14.0000          0          0     0.6000    -0.6000

>> B = [0;0;0;0;0;0;0;2]

     0
     0
     0
     0
     0
     0
     0
     2

>> C = [0,0,0,1,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

Zero/pole/gain:
           2 (s^2 + 0.3515s + 8.201) (s^2 + 1.2s + 28) (s^2 + 2.049s + 47.8)
-------------------------------------------------------------------------------
(s^2 + 0.07237s + 1.689) (s^2 + 0.6s + 14) (s^2 + 1.408s + 32.86) (s^2 + 2.119s + 49.45)

>>
```

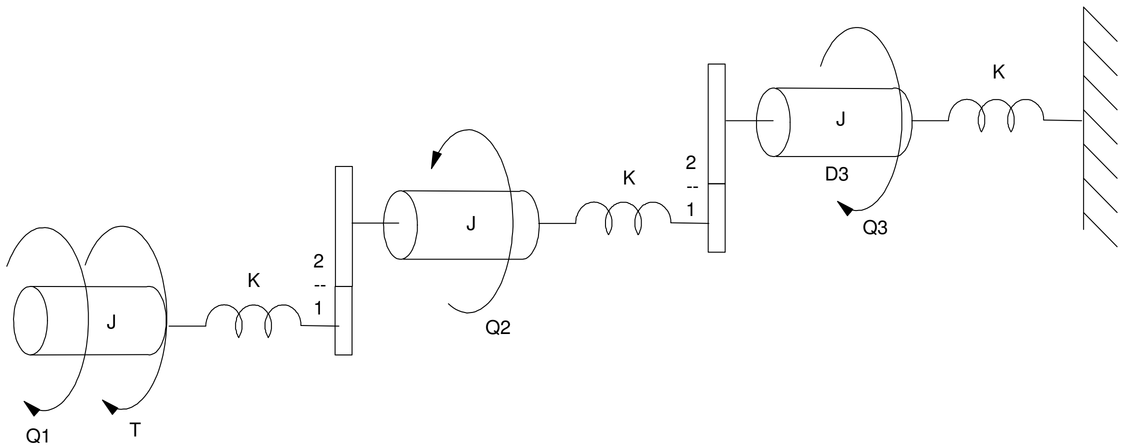
```
>> t = [0:0.01:50]';
>> y = step(G,t);
>> plot(t,y);
```



Rotational Systems

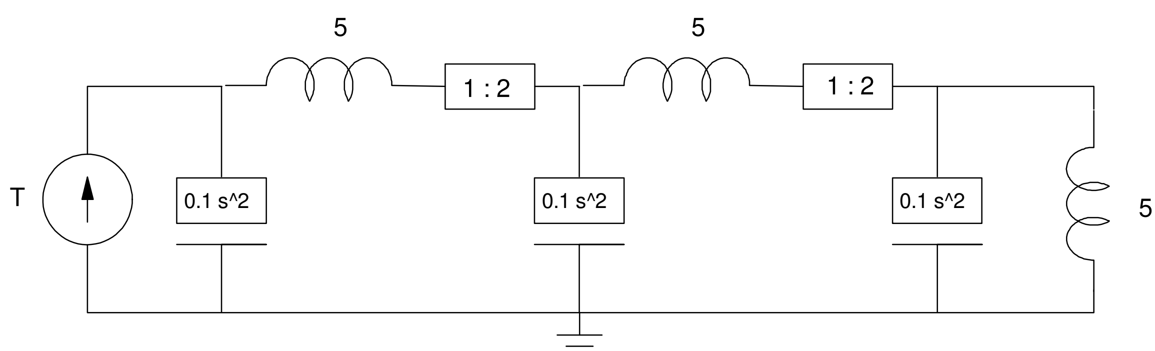
3) Draw the circuit equivalent for the following rotational system.

- Express the dynamics in state-space form
- Find the transfer function from T to Q1
- Plot the step response from T to Q1

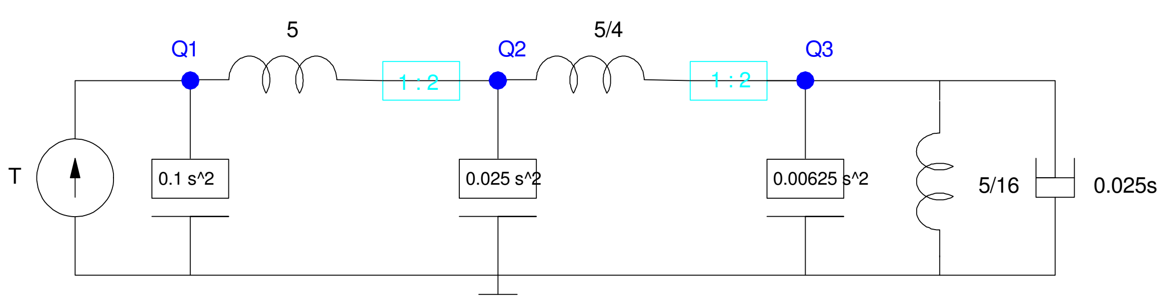


Problem 3: J = 0.1 Kg m / s^2. K = 5 Nm/rad, D3 = 0.4 Nms/ rad

First, draw the circuit equivalent



Remove the gears (adding in D3 which I forgot...)



Write the node equations

$$(0.1s^2+5)\theta_1-5\theta_2=T$$

$$\left(0.025s^2+5+\tfrac{5}{4}\right)\theta_2-5\theta_1-\tfrac{5}{4}\theta_3=0$$

$$\left(0.00625s^2+\tfrac{5}{16}+\tfrac{5}{4}+0.025s\right)\theta_3-\tfrac{5}{4}\theta_2=0$$

Solve for the highest derivative

$$s^2\theta_1=-50\theta_1+50\theta_2+10T$$

$$s^2\theta_2=-250\theta_2+200\theta_1+50\theta_3$$

$$s^2\theta_3=-250\theta_3-4s\theta_3+200\theta_2$$

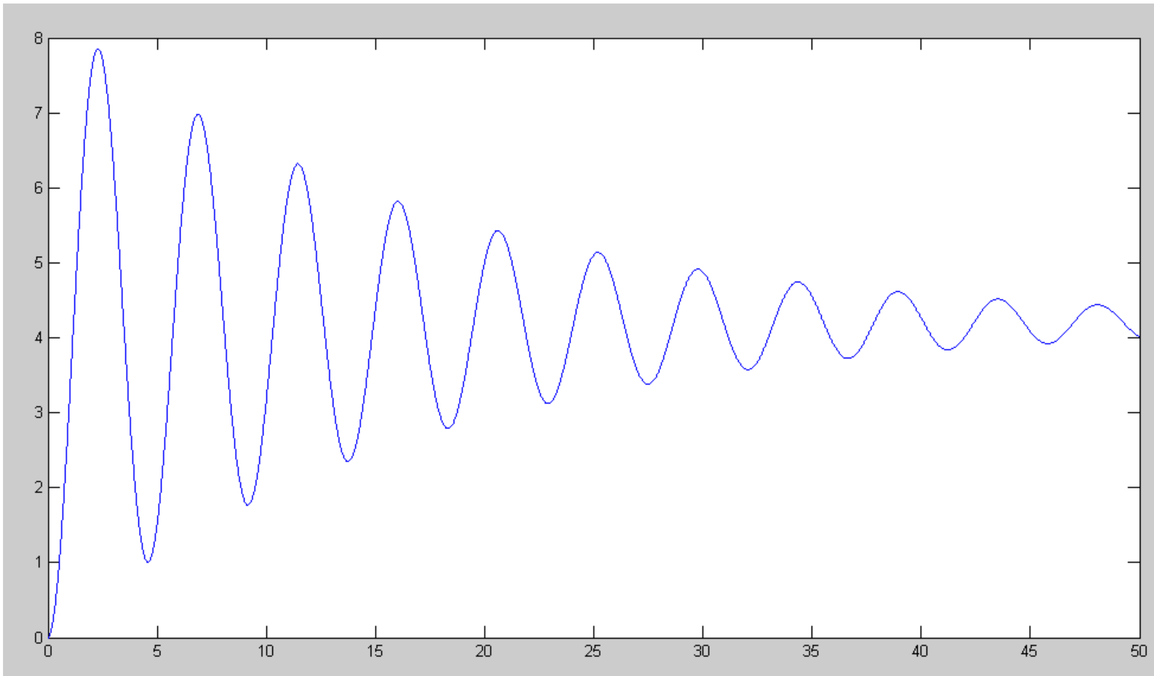
Place in matrtix form

$$s\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\\s\theta_1\\s\theta_2\\s\theta_3\end{bmatrix}=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-50&50&0&0&0&0\\200&-250&50&0&0&0\\0&200&-250&0&0&-4\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\\s\theta_1\\s\theta_2\\s\theta_3\end{bmatrix}+\begin{bmatrix}0\\0\\0\\10\\0\\0\end{bmatrix}T$$

Solve in Matlab

```
>> a11 = zeros(3,3);
>> a12 = eye(3,3);
>> a21 = [-50,50,0;200,-250,50;0,200,-250];
>> a22 = [0,0,0;0,0,0;0,0,-4];
>> A = [a11,a12 ; a21, a22]

A =

     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
   -50    50     0     0     0     0
   200  -250    50     0     0     0
     0   200  -250     0     0    -4

>> B = [0;0;0;10;0;0];
>> C = [1,0,0,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
>> zpk(G)

          10 (s^2 + 2.042s + 153.1) (s^2 + 1.958s + 342.9)
------------------------------------------------------------------
(s^2 + 0.1187s + 1.884) (s^2 + 2.351s + 184.5) (s^2 + 1.53s + 359.6)

>>
```

```
>> t = [0:0.01:50]';
>> y = step(G,t);
>> plot(t,y);
```



Motors

4) Find the transfer function for the following DC servo motor

http://www.regalbeloit.com.my/pdfs/motors/TA_Servo_DC_Baldor.pdf

Baldor MT4525-BTYCN: 1100W DC Servo Motor (1.5hp)

- $649ea
- Kt = 0.61 Nm/A
- Ra = 1.99 Ohms
- La = 9.0mH
- J = 0.0016 kg m2
- Mechanical time constant = 8.43ms
- Electrical time constant = 4.520ms
- Weight 7.0kg

Mechanical pole is 1 / 8.43ms = 118.62

$$(Js + D) = J(s + 118.62)$$

$$\frac{D}{J} = 118.62$$

D = 0.1879 N / rad/sec

Electrical pole is 1 / 4.520ms = 221.23

$$(Ls + R) = L(s + 221.23)$$

$$\frac{R}{L} = 221.23$$

$$L = 9.00mH$$

So

$$\omega = \left(\frac{K_t}{(Js+D)(Ls+R)+K_t^2}\right) V_a$$

```
Kt = 0.61;
R = 1.99;
L = 0.009;
J = 0.0016;
D = 118.62 * J;
G = tf(Kt, [J*L, J*R + L*D, D*R + Kt^2]);
zpk(G)

        42361.1111
--------------------------
(s^2 + 339.7s + 5.207e004)
```

note: It seems odd, but the motor has a complex pole when not loaded.

5) Assume this motor is used to power an electric bicycle at 20mph

- Motor speed @ 20mph = 2500 rpm
- Gear (wheel) used to convert 2500 rpm to 20mph
- Bicycle weight = 100kg

What is the gear reduction (wheel diameter) to convert 2500rpm to 20mph?

Converting to metric

2500rpm = 261.799 rad/sec

20mph = 8.9408 m/s

For a gear

$$d = r\theta$$

taking the derivative

$$v = r\omega$$

$$8.9409 m/s = r \cdot 261.799 \frac{rad}{\sec}$$

$$r = 0.03415 m$$

$$r = 34.15 mm$$

(about a 3" diameter wheel, similar to what drives a skateboard)

What is the inertia relative to the DC servo motor (bring the 100kg mass back to the motor through a gear)

$$J = \left(\frac{r}{1}\right)^2 \cdot 100 kg$$

$$J = 0.1166\, kg\, m^2$$

What is the transfer function (dynamics) for the bicycle / servo motor combination?

```
Kt = 0.61;
R = 1.99;
L = 0.009;
J = 0.0016;
D = 118.62 * J;
J = J + 0.1166;
G = tf(Kt, [J*L, J*R + L*D, D*R + Kt^2]);
zpk(G)

     573.4161
-------------------
(s+219.5) (s+3.211)
```

Note that the complex pole becomes real when you add a load. Motors are designed to drive something.