Homework #9: ECE 461/661

z-Transforms, s to z conversion, Root Locus in the z-Domain. Due Monday, November 9th

z-Transforms

1) Determine the difference equation that relates X and Y

$$Y = \left(\frac{0.005z}{(z-0.95)(z-0.9)(z-0.5)} \right) X$$

Cross multiply

$$(z-0.95)(z-0.9)(z-0.5)Y = (0.005z)X$$

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-->poly([0.95,0.9,0.5])

 ans  =    1.  - 2.35    1.78  - 0.4275
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$$(z^3 - 2.35z^2 + 1.78z - 0.4275)Y = (0.005z)X$$

'zY' means 'the next value of y(k)' or 'y(k+1)'

y(k+3) - 2.35 y(k+2) + 1.78 y(k+1) - 0.4275 y(k) = 0.005 x(k+1)

or using a change in variable

y(k) - 2.35 y(k-1) + 1.78 y(k-2) - 0.4275 y(k-3) = 0.005 x(k-2)

Either answer is correct

2) Determine y(k) assuming

$$Y = \left(\frac{0.005z}{(z-0.95)(z-0.9)(z-0.5)} \right) X \qquad x(k) = u(k)$$

Substitute the z-transform for x(k)

$$Y = \left(\frac{0.005z}{(z-0.95)(z-0.9)(z-0.5)} \right) \left(\frac{z}{z-1} \right)$$

Pull out a z (we'll need this later)

$$Y = \left(\frac{0.005z}{(z-1)(z-0.95)(z-0.9)(z-0.5)} \right) z$$

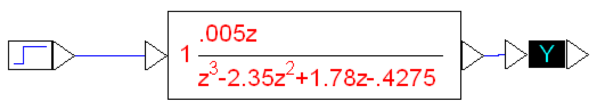
Do partial fractions

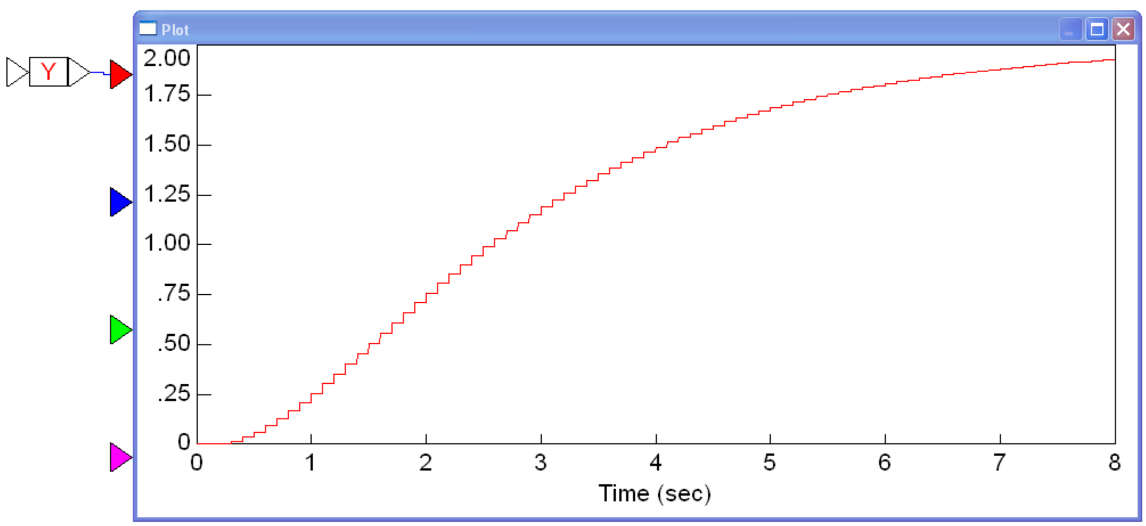
$$Y = \left(\left(\frac{2}{z-1} \right) + \left(\frac{-4.4444}{z-0.95} \right) + \left(\frac{2.25}{z-0.9} \right) + \left(\frac{-0.0278}{z-0.5} \right) \right) z$$

$$Y = \left(\frac{2z}{z-1} \right) + \left(\frac{-4.4444z}{z-0.95} \right) + \left(\frac{2.25z}{z-0.9} \right) + \left(\frac{-0.0278z}{z-0.5} \right)$$

Take the inverse z-transform

$$y(k) = \left(2 - 4.4444(0.95)^k + 2.25(0.9)^k - 0.0278(0.5)^k \right) u(k)$$





3) Determine y(t) assuming

$$Y = \left(\frac{0.005z}{(z^2-1.6z+0.68)(z-0.5)}\right)X \qquad x(k) = u(k)$$

Plug in the z-transform for x(k). Factor the roots

$$Y = \left(\frac{0..005z}{(z-0.8+j0.2)(z-0.8-j0.2)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

Pull out a z

$$Y = \left(\frac{0..005z}{(z-1)(z-0.8+j0.2)(z-0.8-j0.2)(z-0.5)}\right)z$$

Do partial fractions

$$Y = \left(\left(\frac{0.125}{z-1}\right) + \left(\frac{-0.0433-j0.0913}{z-0.8+j0.2}\right) + \left(\frac{-0.0433+j0.0913}{z-0.8-j0.2}\right) + \left(\frac{-0.0385}{z-0.5}\right)\right)z$$

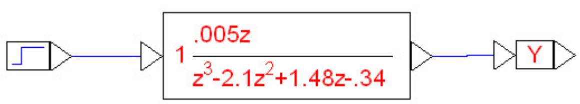
$$Y = \left(\frac{0.125z}{z-1}\right) + \left(\frac{-0.0433-j0.0913}{z-0.8+j0.2}\right)z + \left(\frac{-0.0433+j0.0913}{z-0.8-j0.2}\right)z + \left(\frac{-0.0385z}{z-0.5}\right)$$

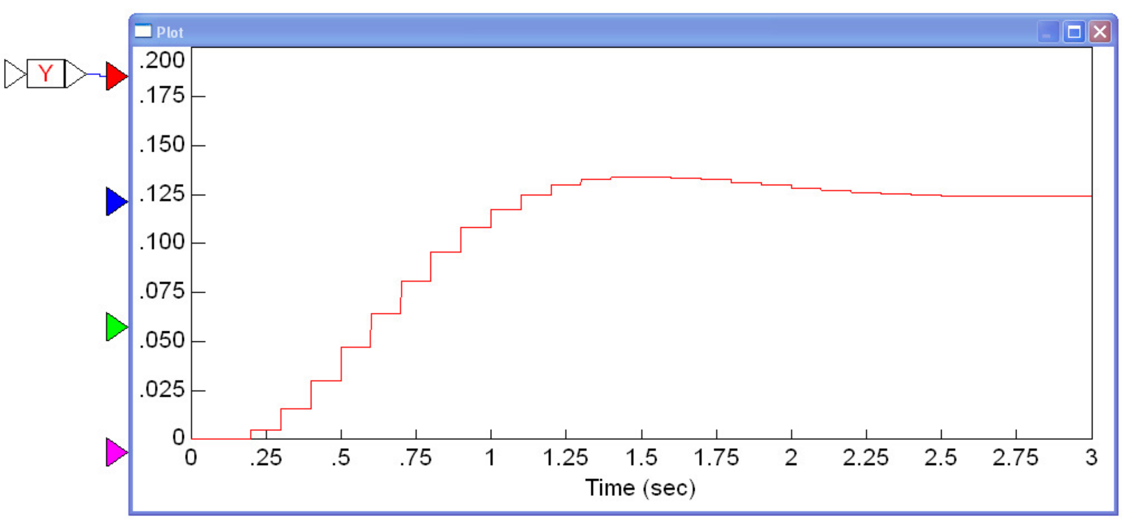
polar form actually works better with z transforms

$$Y = \left(\frac{0.125z}{z-1}\right) + \left(\frac{0.1011\angle -115.3^0}{z-0.8246\angle -14.04^0}\right)z + \left(\frac{0.1011\angle 115.3^0}{z-0.8246\angle 14.04^0}\right)z + \left(\frac{-0.0385z}{z-0.5}\right)$$

Take the inverse z-transform

$$y(k) = \left(0.125 + 0.2022(0.8246)^k \cos\left(14.04^0 \cdot k + 115.3^0\right) - 0.0385(0.5)^k\right)u(k)$$





s to z conversion

3) Determine the discrete-time equivalent of G(s). Assume T = 0.5 second

$$G(s) = \left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)}\right)$$

The converstion from the s to z plane is

$$z = e^{sT}$$

s = -0.1617	z = 0.9223
s = -1.04	z = 0.5945
s = -2.719	z = 0.2568
s = -5.05	z = 0.0801

giving

$$G(z) \approx \left(\frac{k}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)}\right)$$

Matching the DC gain

$$\left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)}\right)_{s=0} = 0.6248$$

$$\left(\frac{k}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)}\right)_{z=1} = 0.6248$$

$$k = 0.0135$$

$$G(z) \approx \left(\frac{0.0135}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)}\right)$$

(optional) G(z) has too much delay. You can fix this by adding zeros at z=0.

To determine how many zeros you need, match the phase shift at s = j0.1 rad/sec

$$\left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)}\right)_{s=j0.1} = 0.5285\angle-40.4669^0$$

$$z = e^{sT} = e^{(j0.1)(0.5)} = e^{j0.05} = 1\angle 0.05$$

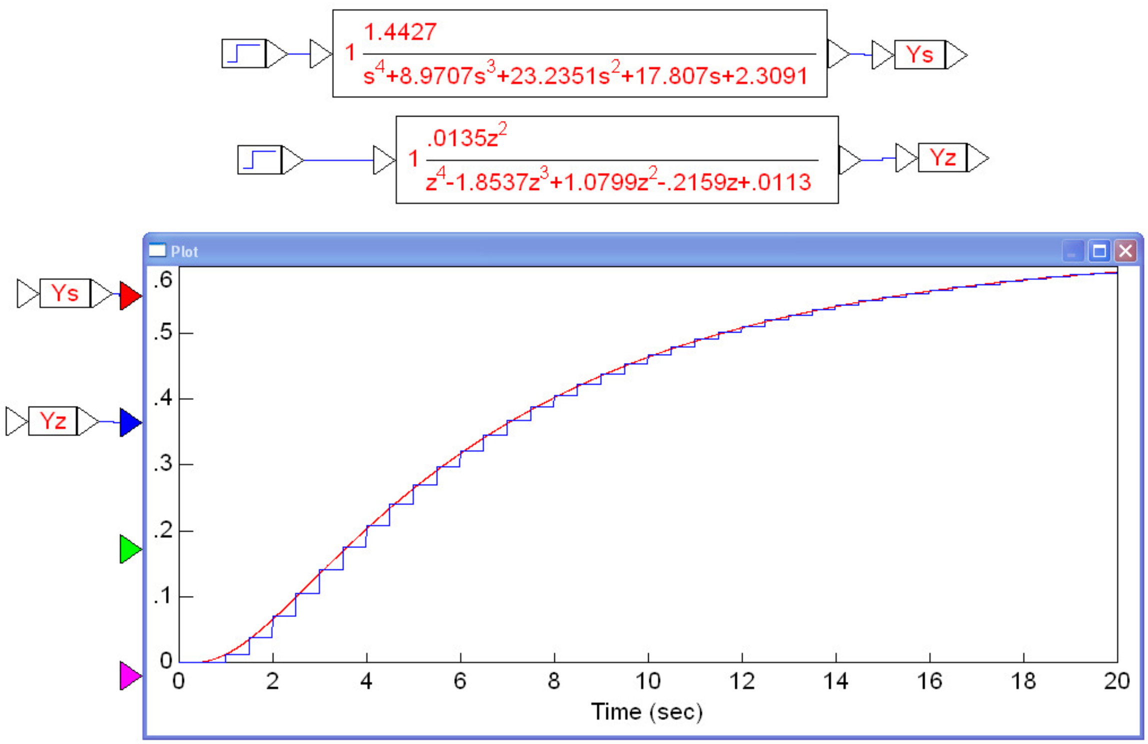
$$\left(\frac{0.0135}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)}\right)_{z=1\angle 0.05} = 0.5304\angle-47.1903^0$$

for a difference of 6.7234 degrees. Each zero at z=0 adds 0.05 radians = 2.86 degrees. The number of zeros you need to make the phase match is

$$n = \left(\frac{6.7234^0}{2.86^0}\right) = 2.34$$

Round to n = 2

$$G(z) \approx \left(\frac{0.0135z^2}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)}\right)$$



4) Determine the discrete-time equivalent of G(s). Assume T = 0.1 second

$$G(s) = \left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)} \right)$$

Converting from the s-plane to the z-plane

s = -0.1617 z = 0.9840

s = -1.04 z = 0.9012

s = -2.719 z = 0.7619

s = -5.05 z = 0.6035

giving

$$G(z) \approx \left(\frac{k}{(z-0.9840)(z-0.9012)(z-0.7619)(z-0.6035)} \right)$$

Pick k to match the DC gain

$$G(s=0) = 0.6248$$

$$\left(\frac{k}{(z-0.9840)(z-0.9012)(z-0.7619)(z-0.6035)} \right)_{z=1} = 0.6248$$

$$k = 0.0000932$$

$$G(z) \approx \left(\frac{0.0000932}{(z-0.9840)(z-0.9012)(z-0.7619)(z-0.6035)} \right)$$

(optional) Add zeros at z=0 to match the phase shift at s = j0.1

$$z = e^{sT} = e^{(j0.1)(0.1)} = e^{j0.01} = 1\angle 0.01$$

$$\left(\frac{0.0000932}{(z-0.9840)(z-0.9012)(z-0.7619)(z-0.6035)} \right)_{z=1\angle 0.01} = 0.5279\angle 41.718^0$$

$$\left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)} \right)_{s=j0.1} = 0.5285\angle -40.4669^0$$

for a difference of 1.2513 degrees (0.0218 radian). Each zero at z=0 adds 0.01 radian, meaning you need 2.18 zeros. Let n=2

$$G(z) \approx \left(\frac{0.0000932z^2}{(z-0.9840)(z-0.9012)(z-0.7619)(z-0.6035)} \right)$$

Note: If you change the sampling rate, G(z) changes.

- ***Any compensator you designed for G(z) at T = 0.5 won't work for T = 0.1.***
- ***Changing the sampling rate is a big deal***

Root Locus in the z-Domain

5) Assume T = 0.5 seconds.

$$G(s) = \left(\frac{1.4427}{(s+0.1617)(s+1.04)(s+2.719)(s+5.05)} \right)$$

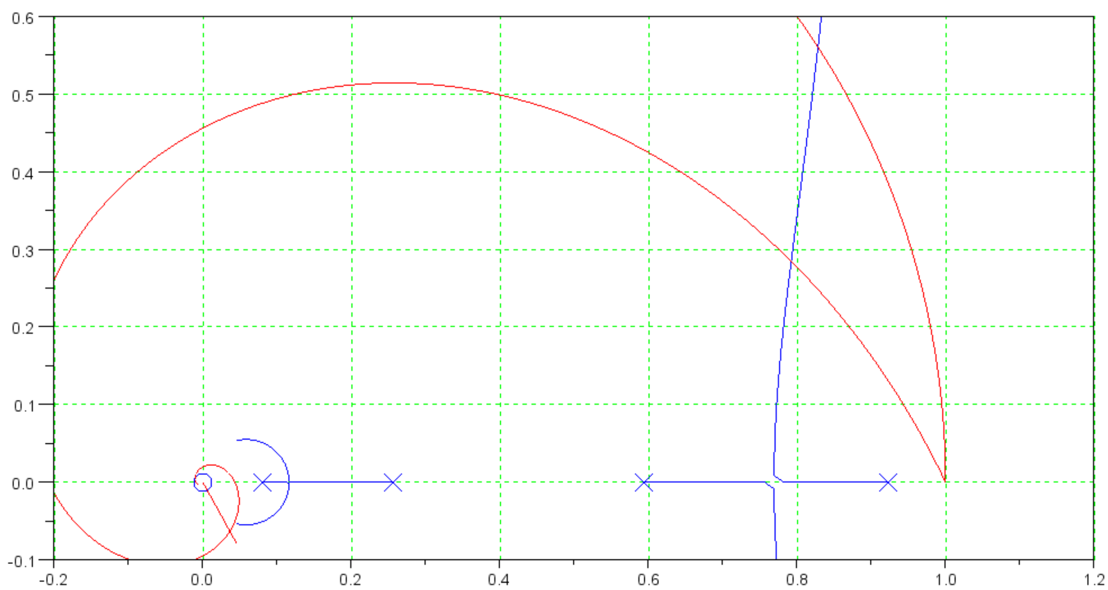
Draw the root locus for G(z)

From problem #3

$$G(z) \approx \left(\frac{0.0135z^2}{(z-0.9223)(z-0.5945)(z-0.2568)(z-0.0801)} \right)$$

Drawing the root locus in Matlab

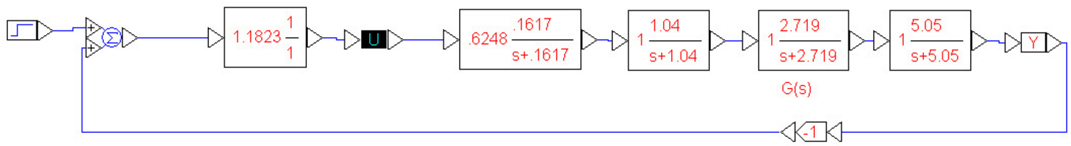
```
G = zpk([0,0],[0.9223,0.5945,0.2568,0.0801],0.0135);
k = logspace(-2,2,1000)';

rlocus(G,k);
hold on

s = [0:0.01:10] * (-1+j*2);
T = 0.5;
z = exp(s*T);
plot(real(z),imag(z),'r');

s = [0:0.01:10] * (0+j*1);
z = exp(s*T);
plot(real(z),imag(z),'r');
```

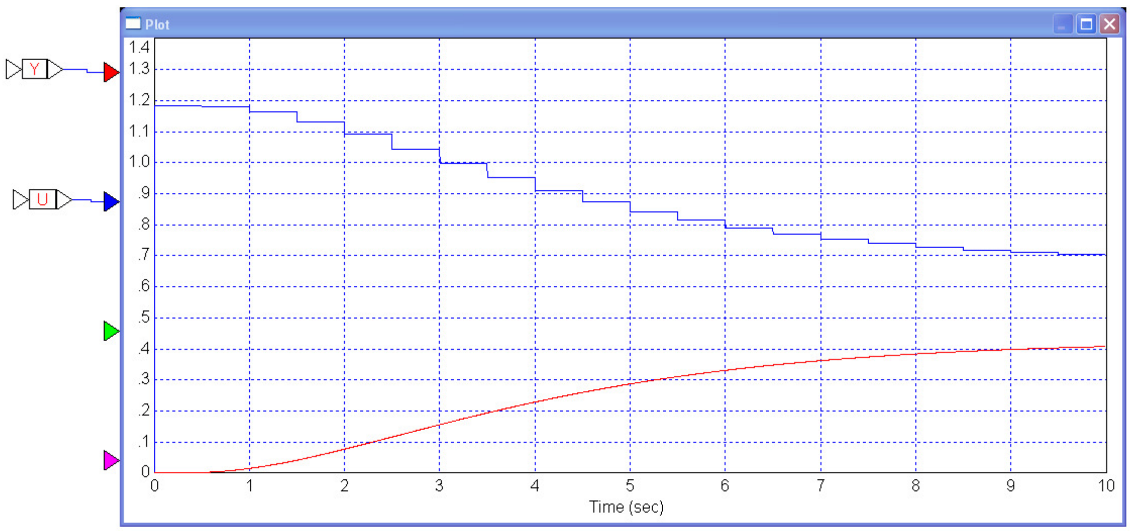


6) Find k for no overshoot in the step response

- Simulate the step response of the closed-loop system

```
G = zpk([0,0],[0.9223,0.5945,0.2568,0.0801],0.0135);

z = 0.77
evalfr(G,z)

  - 0.8457956

k = 1/abs(ans)

 k  =    1.1823188
```

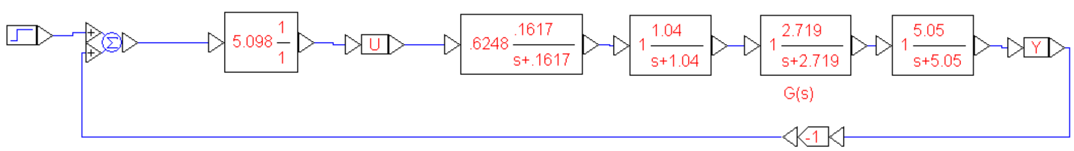


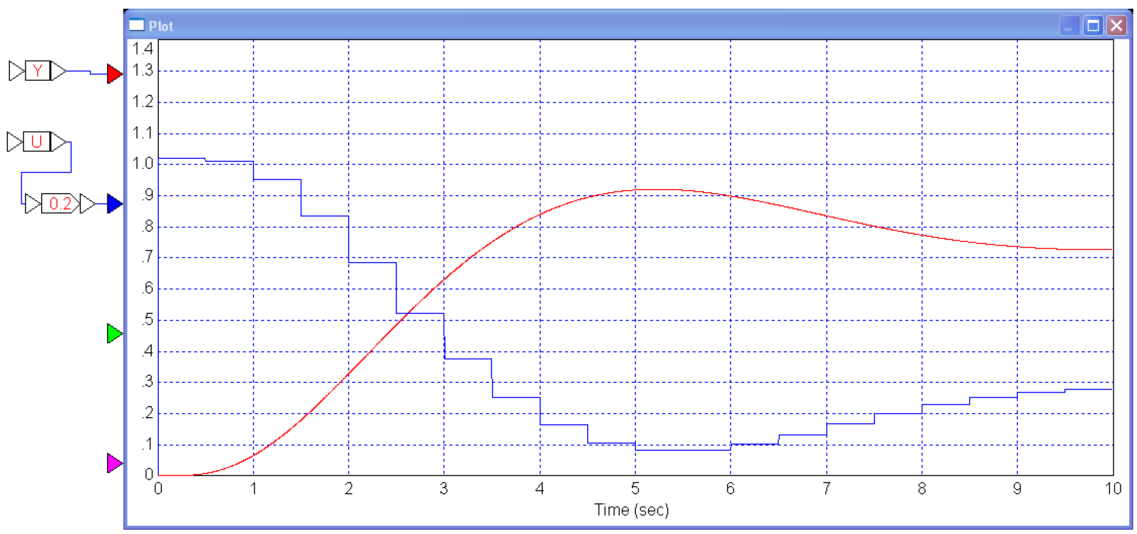


7) Find k for a damping ratio of 0.4559 (20% overshoot)

- Simulate the step response of the closed-loop system

```
G = zpk([0,0],[0.9223,0.5945,0.2568,0.0801],0.0135);

z = 0.78 + j*0.28;
evalfr(G,z)

  - 0.1960269 + 0.0132974i

k = 1/abs(ans)

 k  =   5.0896433
```





8) Find k for a damping ratio of 0.000

- Simulate the step response of the closed-loop system

```
G = zpk([0,0],[0.9223,0.5945,0.2568,0.0801],0.0135);

z = 0.82 + j*0.56;
evalfr(G,z)

  - 0.0525318 + 0.0015737i

k = 1/abs(ans)

 k  =  19.027567
```